Completeness of non-normalizable modes

2006 J. Phys. A: Math. Gen. 39 13783

(http://iopscience.iop.org/0305-4470/39/44/012)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 13783–13806 doi:10.1088/0305-4470/39/44/012

Completeness of non-normalizable modes

Philip D Mannheim and Ionel Simbotin

Department of Physics, University of Connecticut, Storrs, CT 06269, USA

E-mail: philip.mannheim@uconn.edu and simbotin@phys.uconn.edu

Received 13 July 2006
Published 17 October 2006
Online at stacks.iop.org/JPhysA/39/13783

Abstract

We establish the completeness of some characteristic sets of non-normalizable modes by constructing fully localized square steps out of them, with each such construction expressly displaying the Gibbs phenomenon associated with trying to use a complete basis of modes to fit functions with discontinuous edges. As well as being of interest in and of itself, our study is also of interest to the recently introduced large extra dimension brane-localized gravity program of Randall and Sundrum, since the particular non-normalizable mode bases that we consider (specifically the irregular Bessel functions and the associated Legendre functions of the second kind) are associated with the tensor gravitational fluctuations which occur in those specific brane worlds in which the embedding of a maximally four-symmetric brane in a five-dimensional anti-de Sitter bulk leads to a warp factor which is divergent. Since the brane-world massless four-dimensional graviton has a divergent wavefunction in these particular cases, its resulting lack of normalizability is thus not seen to be any impediment to its belonging to a complete basis of modes, and consequently its lack of normalizability should not be seen as a criterion for not including it in the spectrum of observable modes. Moreover, because the divergent modes we consider form complete bases, we can even construct propagators out of them in which these modes appear as poles with residues which are expressly finite. Thus, even though normalizable modes appear in propagators with residues which are given as their finite normalization constants, non-normalizable modes can just as equally appear in propagators with finite residues too—it is just that such residues will not be associated with bilinear integrals of the modes.

PACS numbers: 02.30.Gp, 03.65.Ge, 04.50.Th, 11.25.−w

1. Introduction

In constructing complete bases of mode solutions to wave equations it is very convenient to work with modes which are normalizable since they obey a closure relation. Specifically, if one has some complete orthonormal basis of modes $f_m(w)$ with eigenvalues labelled by m and orthonormality relation

$$\int_{-\infty}^{\infty} \mathrm{d}w\, \mathrm{e}^{-2A(w)} f_m(w) f_{m'}(w) = \delta_{m,m'}, \tag{1}$$

where $\mathrm{e}^{-2A(w)}$ is an appropriate normalization measure, the completeness of the basis will then require that any localized function be expandable in terms of the basis modes as

$$\psi(w) = \sum_m a_m f_m(w) \tag{2}$$

with coefficients which are given as

$$a_m = \int_{-\infty}^{\infty} \mathrm{d}w\, \mathrm{e}^{-2A(w)} \psi(w) f_m(w). \tag{3}$$

With insertion of these coefficients back into equation (2) yielding

$$\begin{aligned} \psi(w) &= \int_{-\infty}^{\infty} \mathrm{d}w' \delta(w - w')\psi(w') \\ &= \sum_m \int_{-\infty}^{\infty} \mathrm{d}w'\, \mathrm{e}^{-2A(w')} \psi(w') f_m(w') f_m(w), \end{aligned} \tag{4}$$

the arbitrariness of the choice of $\psi(w)$ will then require that the basis modes obey a closure relation of the form

$$\sum_m f_m(w') f_m(w) = \mathrm{e}^{2A(w)} \delta(w - w'). \tag{5}$$

With equation (5) being recognized as being a special case of equation (2) (namely the expansion of the extremely localized $\delta(w)$ in a complete basis of $f_m(w)$ with coefficients $a_m = f_m(0)$), the notions of completeness and closure are often treated interchangeably in the literature, with equation (5) not only often being referred to as being a completeness relation, but with it even being regarded as being an essential requirement for a basis to be complete in the first place.

It is the purpose of this paper to show that this need not in fact be the case and that modes can be complete even when they do not obey equation (5) at all. Indeed, the steps which lead from equation (1) to equation (5) only hold when the basis is one for which the integrals on the left-hand side of equation (1) do in fact exist. With both equations (1) and (5) involving bilinear functions of the basis modes, but with equation (2) only being a linear function of the modes, it is still possible for the summation in equation (2) to be well defined even when the bilinear expressions which appear in equations (1) and (5) are not. Moreover, the wave equations for which $f_m(w)$ are the mode solutions are themselves only linear functions of $f_m(w)$, and it should thus be immaterial to the completeness of their solutions as to whether or not bilinear integrals of the modes exist. In general, then completeness of a basis has to be understood as being the requirement that for localized functions $\psi(w)$ there exists an expansion of the form of equation (2) with finite coefficients a_m regardless of whether or not the integrals on the left-hand side of equation (1) actually exist. Non-normalizable modes whose behaviour is so bad as to cause these bilinear integrals to diverge can still be complete in the sense of equation (2), with the a_m coefficients being such as to lead to total destructive interference between $f_m(w)$ in the regions where $f_m(w)$ diverge. It is thus equation (2) which has to be recognized as being the general statement of completeness, and in this paper we shall confirm this by explicitly constructing localized square steps as sums over some characteristic bases of divergent modes. While the existence or not of the normalization integrals of equation (1) is immaterial to a differential wave equation, if the solutions to the wave equation are required to belong to a Hilbert space one can restrict to square integrable functions alone, though otherwise there is no reason to discard any non-normalizable solutions[1]. Since wave equations in classical physics do not act in a Hilbert space, in classical physics one is not free to discard non-normalizable modes, and since classical physics wave equations play a prominent role in classical gravity where they are associated with classical gravitational fluctuations around classical gravity backgrounds, it is to classical gravity that we shall look for examples in which to test whether non-normalizable modes can be complete.

2. Wave equations for gravitational fluctuations

The wave equations we shall explicitly explore are associated with the recently introduced brane-localized gravity program of Randall and Sundrum [2, 3]. As introduced, the brane gravity program provides for the possibility that our four-dimensional universe could be embedded in some infinitely sized bulk space and yet not conflict with the fact that there is no apparent sign of any such higher dimensional bulk. Specifically, by taking the higher dimensional bulk to possess a very special geometry, namely the five-dimensional anti-de Sitter geometry AdS_5, and by taking our four-dimensional universe to be a brane (i.e. membrane) embedded in it, Randall and Sundrum found that under certain circumstances it was then possible for gravitational signals to localize around the brane and not penetrate very far into the bulk, with AdS_5 acting as a sort of refractive medium which rapidly attenuates any signals which try to propagate in it. Within the Randall–Sundrum brane world there are six fully soluble set-ups (technically AdS_5 bulks with embedded Minkowski, de Sitter or anti-de Sitter branes each with either positive or negative tension λ—to be referred to as the $M_4^{\pm}$, $\mathrm{dS}_4^{\pm}$ and $\mathrm{AdS}_4^{\pm}$ brane worlds in the following), with all six of them having backgrounds which can be described by the generic five-dimensional metric

$$\mathrm{d}s^2 = \mathrm{d}w^2 + \mathrm{e}^{2A(|w|)} q_{\mu\nu}(x^\lambda)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{6}$$

where the w-independent $q_{\mu\nu}$ is the four-dimensional metric and the so-called warp factor $\mathrm{e}^{2A(|w|)}$ is taken to be a function of $|w|$ where w is the fifth coordinate. With the curvature of AdS_5 being taken to be given as $-b^2$, in the various cases the explicit background metrics are given as

$$\mathrm{d}s^2\big(M_4^{\pm}\big) = \mathrm{d}w^2 + \mathrm{e}^{-2\epsilon(\lambda)b|w|}[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}t^2], \tag{7}$$

$$\mathrm{d}s^2\big(\mathrm{dS}_4^{\pm}\big) = \mathrm{d}w^2 + \frac{H^2}{b^2}\sinh^2\left[\operatorname{arcsinh}\left(\frac{b}{H}\right) - \epsilon(\lambda)b|w|\right][\mathrm{e}^{2Ht}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) - \mathrm{d}t^2], \tag{8}$$

and

$$\mathrm{d}s^2\big(\mathrm{AdS}_4^{\pm}\big) = \mathrm{d}w^2 + \frac{H^2}{b^2}\cosh^2\left[\operatorname{arccosh}\left(\frac{b}{H}\right) - \epsilon(\lambda)b|w|\right][\mathrm{d}x^2 + \mathrm{e}^{2Hx}(\mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}t^2)], \tag{9}$$

[1] Even in quantum mechanics we note that the Schrödinger equation $H|\psi\rangle = E|\psi\rangle$ is an operator equation which acts linearly on the ket vector $|\psi\rangle$, with its existence being independent of what particular dual vector bra $\langle\psi|$ might be used to construct the bilinear norm $\langle\psi|\psi\rangle$. There is thus freedom available in choosing the dual space vectors, with choices for them other than simply as the conjugates of the kets having been found to lead to a sensible probability interpretation in the case of theories with a non-Hermitian potential (the first part of [1]) or an indefinite metric (the fourth-order oscillator theory discussed in the second part of [1]). Even in quantum mechanics then, imposing the finiteness of the $\langle\psi|\psi\rangle$ norm is not the most general requirement that one can consider.

where $\epsilon(\lambda)$ is the sign of λ. (The $M_4^{\pm}$ background metrics are given in [2, 3], the $\mathrm{dS}_4^{\pm}$ background metrics are given in [4, 5] and the $\mathrm{AdS}_4^{\pm}$ background metrics are given in [4].)

For the brane world the gravitational fluctuations around these six backgrounds are most readily treated in the axial gauge where the transverse-traceless tensor fluctuation modes $h_{\mu\nu}^{\mathrm{TT}}$ then all obey the generic wave equation (see, e.g., [6] where full derivations and relevant citations are given)

$$\left[\frac{\partial^2}{\partial|w|^2} - 4\left(\frac{\mathrm{d}A}{\mathrm{d}|w|}\right)^2 + \mathrm{e}^{-2A}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\right]h_{\mu\nu}^{\mathrm{TT}} = 0, \tag{10}$$

as subject to the constraint (technically the Israel junction condition)

$$\delta(w)\left[\frac{\partial}{\partial|w|} - 2\frac{\mathrm{d}A}{\mathrm{d}|w|}\right]h_{\mu\nu}^{\mathrm{TT}} = 0 \tag{11}$$

at a brane which is located at $w = 0$. In equation (10), the tildes in $\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha$ indicate that these particular covariant derivatives are to be evaluated in the geometry associated with the four-dimensional $q_{\mu\nu}$. And with the four-dimensional sector of the theory being separable according to

$$[\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha - 2kH^2]h_{\mu\nu}^{\mathrm{TT}} = m^2 h_{\mu\nu}^{\mathrm{TT}}, \tag{12}$$

as defined here so that tensor fluctuations with $m^2 = 0$ propagate on the appropriate dS_4, M_4 or AdS_4 lightcones ($k = 1, 0, -1$, respectively), a separation of the modes into the form $h_{\mu\nu}^{\mathrm{TT}} = f_m(|w|)e_{\mu\nu}(x^\lambda, m)$ then requires that $f_m(|w|)$ obey

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}|w|^2} - 4\left(\frac{\mathrm{d}A}{\mathrm{d}|w|}\right)^2 - 2\left(\frac{\mathrm{d}^2A}{\mathrm{d}|w|^2}\right) + \mathrm{e}^{-2A}m^2\right]f_m(|w|) = 0 \tag{13}$$

(in each of the six background cases of interest to us the identity $\mathrm{d}^2A/\mathrm{d}|w|^2 = -kH^2\,\mathrm{e}^{-2A}$ holds), as subject to the constraint

$$\delta(w)\left[\frac{\mathrm{d}}{\mathrm{d}|w|} - 2\frac{\mathrm{d}A}{\mathrm{d}|w|}\right]f_m(|w|) = 0. \tag{14}$$

Our task is thus to explore the completeness of solutions to equations (13) and (14), and a reader unfamiliar with the physics of the brane world can start at this point as none of the analysis which ensues will depend on how equations (13) and (14) were first arrived at. What will matter in the following is only that these equations admit of exact solutions, solutions whose large $|w|$ behaviour can then explicitly be monitored.

Before actually identifying explicit solutions to equations (13) and (14) for the specific choices of A and $\epsilon(\lambda)$ of interest, we note that via manipulation of equation (13) we find that every pair of its solutions have to obey

$$\mathrm{e}^{-2A}\left(m_1^2 - m_2^2\right)f_{m_1}f_{m_2} = \frac{\mathrm{d}}{\mathrm{d}|w|}\left[f_{m_1}\left(\frac{\mathrm{d}}{\mathrm{d}|w|} - 2\frac{\mathrm{d}A}{\mathrm{d}|w|}\right)f_{m_2} - f_{m_2}\left(\frac{\mathrm{d}}{\mathrm{d}|w|} - 2\frac{\mathrm{d}A}{\mathrm{d}|w|}\right)f_{m_1}\right], \tag{15}$$

which with equation (14) then requires the modes to obey

$$\left(m_1^2 - m_2^2\right)\int_0^\infty \mathrm{d}|w|\,\mathrm{e}^{-2A}f_{m_1}f_{m_2} = \lim_{|w|\to\infty}\left[f_{m_1}\left(\frac{\mathrm{d}}{\mathrm{d}|w|} - 2\frac{\mathrm{d}A}{\mathrm{d}|w|}\right)f_{m_2} - f_{m_2}\left(\frac{\mathrm{d}}{\mathrm{d}|w|} - 2\frac{\mathrm{d}A}{\mathrm{d}|w|}\right)f_{m_1}\right]. \tag{16}$$

Orthogonality of modes with different separation constants is thus achieved when the modes are well-enough behaved at $|w| = \infty$ to cause the right-hand side of equation (16) to vanish (with the orthogonality measure then being precisely the one we introduced in equation (1)), with modes which diverge badly enough at infinity causing the integral on the left-hand side to not exist. While one could now proceed to determine the mode solutions and identify for which particular ones the integral on the left-hand side of equation (16) converges or diverges, before doing so it is instructive to recall that via a sequence of transformations it is possible to bring equation (13) to a more familiar form. Specifically, if we change variables from w to z by setting $\mathrm{d}z = \mathrm{e}^{-A(w)}\mathrm{d}w$ and define $f_m = \mathrm{e}^{A(z)/2}\hat{f}_m$, $\hat{f}_m$ will then obey [3]

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{9}{4}\left(\frac{\mathrm{d}A}{\mathrm{d}z}\right)^2 + \frac{3}{2}\frac{\mathrm{d}^2 A}{\mathrm{d}z^2} - m^2\right]\hat{f}_m = 0, \tag{17}$$

while at the same time the normalization integral will change as

$$\int_0^\infty \mathrm{d}|w|\,\mathrm{e}^{-2A} f_{m_1}(|w|) f_{m_2}(|w|) \to \int_{z[0]}^{z[\infty]} \mathrm{d}z\,\hat{f}_{m_1}(z)\hat{f}_{m_2}(z). \tag{18}$$

While we thus recognize equation (17) as being in the familiar form of a one-dimensional Schrödinger equation and equation (18) as being in the form of its conventional quantum-mechanical normalization integral, nonetheless, as noted above, since in the cases which are of interest to us here we are not requiring the $\hat{f}_m$ modes to belong to a Hilbert space, we should not discard the non-normalizable solutions to equation (17).[2] And having now recognized the rationale for not discarding non-normalizable solutions, we return to equations (13) and (14) to actually find and then explore them.

3. Completeness tests for the Minkowski brane cases

3.1. Positive tension case

For the M_4^+ case where $A = -b|w|$, the solutions to equation (13) are readily obtained by setting $y = m\,\mathrm{e}^{b|w|}/b$ as this transformation brings equation (13) to the Bessel equation form

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y} + 1 - \frac{4}{y^2}\right] f_m(y) = 0. \tag{19}$$

Mode solutions with any positive m^2 are thus given by

$$f_m(y) = \alpha_m J_2(y) + \beta_m Y_2(y) \tag{20}$$

where α_m and β_m are y-independent coefficients, with those solutions with $m^2 = 0$ being given directly from equation (13) as

$$f_0(y) = \alpha_0\,\mathrm{e}^{-2b|w|} + \beta_0\,\mathrm{e}^{2b|w|}. \tag{21}$$

To satisfy the junction condition of equation (14) then requires that the various mode coefficients obey

$$\alpha_m J_1(m/b) + \beta_m Y_1(m/b) = 0, \qquad \beta_0 = 0, \tag{22}$$

with the continuum of $m^2 > 0$ modes thus satisfying the junction condition via an interplay of the two types of Bessel function, and the $m^2 = 0$ mode $f_0(y) = \alpha_0\,\mathrm{e}^{-2b|w|}$ satisfying it all on

[2] Even in quantum mechanics one does not discard plane wave modes even though they cause the integral on the right-hand side of equation (18) to diverge, since divergent as they may be, one can still construct localized wave packets out of them. In this respect then, the point of this paper will be to construct localized configurations out of basis vectors which diverge even more rapidly than plane waves. And while we shall restrict the study of this paper to the classical-mechanical context, we note that within a quantum-mechanical context such localized configurations could still belong to a Hilbert space even if the basis vectors themselves out of which they are built do not.

its own. In the brane world the $m^2 > 0$ modes are known as the KK (Kaluza–Klein) modes, while the $m^2 = 0$ mode serves as a massless graviton. At large y, these solutions behave as

$$f_m \to \left(\frac{2}{\pi y}\right)^{1/2} [\alpha_m \cos(y - 5\pi/4) + \beta_m \sin(y - 5\pi/4)], \qquad f_0 \to \frac{\alpha_0 m^2}{b^2 y^2}. \tag{23}$$

With all of these modes having wavefunctions which fall very fast in $|w|$ as we go away from the brane, the gravitational fluctuation modes are thus localized around it, this being the key result of [3]. With the measure of the normalization integral being rewriteable as

$$\int_0^\infty \mathrm{d}|w|\, \mathrm{e}^{2b|w|} = b\int_{1/b}^\infty \mathrm{d}x\, x \tag{24}$$

on setting $x = \mathrm{e}^{b|w|}/b$, we see that the massless graviton wavefunction is bound state normalizable and that the KK modes possess the same continuum normalization as flat space Bessel functions. Consequently, the totality of massless graviton plus KK continuum modes is complete in exactly the same way as plane waves, with both of equations (1) and (5) being satisfied (the summation in equation (5) is understood to contain both discrete and continuous indices). While we thus see that there is no need to perform any explicit completeness test for the modes of M_4^+ as everything is standard, a quite different situation will emerge when we consider M_4^-.

3.2. Negative tension case

For the M_4^- case where $A = +b|w|$, the $m^2 > 0$ and the $m^2 = 0$ solutions to equation (13) are given by

$$f_m(y) = \alpha_m J_2(y) + \beta_m Y_2(y), \tag{25}$$

and

$$f_0(y) = \alpha_0\, \mathrm{e}^{-2b|w|} + \beta_0\, \mathrm{e}^{2b|w|}, \tag{26}$$

where now $y = m\,\mathrm{e}^{-b|w|}/b$, while to satisfy the junction condition of equation (14) this time requires

$$\alpha_m J_1(m/b) + \beta_m Y_1(m/b) = 0, \qquad \alpha_0 = 0. \tag{27}$$

Unlike the M_4^+ case this time y goes to zero as $|w|$ goes to infinity, with large $|w|$ asymptotics now being controlled by the behaviour of Bessel functions at small argument rather than large, with the solutions behaving at small y as

$$f_m \to \frac{\alpha_m y^2}{8} - \frac{4\beta_m}{\pi y^2}, \qquad f_0 \to \frac{\beta_0 m^2}{b^2 y^2} \tag{28}$$

($Y_2(y)$ behave irregularly at small argument). With the measure of the normalization integral now being given as

$$\int_0^\infty \mathrm{d}|w|\, \mathrm{e}^{-2b|w|} = b\int_0^{1/b} \mathrm{d}x\, x \tag{29}$$

on setting set $x = \mathrm{e}^{-b|w|}/b$, this time we see that it is only the $J_2(y)$ modes which are normalizable, and that the massless graviton wavefunction and all the $Y_2(y)$ modes are not only non-normalizable, they diverge far too violently to even be plane wave normalizable. In order to be able to satisfy the junction condition of equation (27) with normalizable modes alone, the convergent $J_2(y)$ modes would have to satisfy equation (27) all by themselves, with the modes then needing to obey $J_1(m/b) = 0$. Solutions to this condition exist and are given as the zeros, j_i, of the Bessel function J_1. This set of zeros is discrete and infinite, with the normalizable modes of the M_4^- brane world then being given as modes with masses $m_i = bj_i$. Similarly, the divergent $Y_2(y)$ modes can satisfy the junction condition all on their own if their masses obey $m_i = by_i$, where y_i are the zeros of the Bessel function Y_1, to yield another infinite set of discrete modes. With the divergent massless graviton mode with wavefunction $\beta_0\,\mathrm{e}^{2b|w|}$ also satisfying the junction condition on its own, we thus recognize two classes of basis modes in the M_4^- brane world, the convergent $J_2(j_i\,\mathrm{e}^{-b|w|})$, and the divergent $\mathrm{e}^{2b|w|}$ and $Y_2(y_i\,\mathrm{e}^{-b|w|})$. And while our objective is to apply a completeness test to the divergent mode basis, it will be instructive to actually apply a completeness test to the convergent M_4^- mode basis first.

4. Completeness test for convergent M_4^- modes

To test for completeness of a basis, we need to determine whether it is possible to expand the typical localized square step $V_J = \hat{V}, \alpha \leqslant \mathrm{e}^{-b|w|}/b \leqslant \beta$, $V_J = 0$ otherwise in terms of the modes of the basis, namely we seek to find a set of V_m from which we can reconstruct the square step according to

$$V_J(|w|) = \sum_m V_m J_2(m\,\mathrm{e}^{-b|w|}/b). \tag{30}$$

To determine the needed coefficients V_m, we apply $\int_0^\infty \mathrm{d}|w|\,\mathrm{e}^{-2b|w|} J_2(m\,\mathrm{e}^{-b|w|}/b)$ to equation (30) and use the orthogonality relations that the asymptotically well-behaved $J_2(m\,\mathrm{e}^{-b|w|}/b)$ modes obey. Specifically, with the right-hand side of equation (16) vanishing for these modes, the modes will then obey

$$\int_0^\infty \mathrm{d}|w|\,\mathrm{e}^{-2b|w|} J_2(m\,\mathrm{e}^{-b|w|}/b) J_2(m'\,\mathrm{e}^{-b|w|}/b) = 0 \tag{31}$$

when m is not equal to m', with use of some standard properties of Bessel functions obliging them to obey

$$\begin{aligned}\int_0^\infty \mathrm{d}|w|\,\mathrm{e}^{-2b|w|} J_2^2(m\,\mathrm{e}^{-b|w|}/b) &= b\int_0^{1/b} \mathrm{d}x\, x J_2^2(mx)\\ &= b\frac{x^2}{2}\left[J_2^2(mx) - J_1(mx)J_3(mx)\right]\Big|_0^{1/b} = \frac{J_2^2(m/b)}{2b}\end{aligned} \tag{32}$$

when m and m' are equal and m is such that $J_1(m/b)$ is zero. Armed with equations (31) and (32) we thus find that $V_J(|w|)$ is to be given by

$$V_J(|w|) = \sum_m \frac{2bB_m}{J_2^2(m/b)} J_2(m\,\mathrm{e}^{-b|w|}/b), \tag{33}$$

where the coefficients B_m are given by

$$\begin{aligned}B_m &= \int_0^\infty \mathrm{d}|w|\,\mathrm{e}^{-2b|w|} V_J(|w|) J_2(m\,\mathrm{e}^{-b|w|}/b) = -b\hat{V}\int_\alpha^\beta x\,\mathrm{d}x\, J_2(mx)\\ &= -\frac{b\hat{V}}{m^2}\int_{m\alpha}^{m\beta}[2J_1(x) - xJ_0(x)] = \frac{b\hat{V}}{m^2}[2J_0(x) + xJ_1(x)]|_{m\alpha}^{m\beta}\\ &= \frac{b\hat{V}}{m^2}[2J_0(m\beta) + m\beta J_1(m\beta)] - \frac{b\hat{V}}{m^2}[2J_0(m\alpha) + m\beta J_1(m\alpha)].\end{aligned} \tag{34}$$

With every quantity which appears in equation (33) now being known, $V_J(|w|)$ can readily be plotted, and we display it in figure 1 as evaluated[3] through the use of the first 1000 modes in

[3] While equation (33) is given in closed form, the actual sum over modes is itself done numerically.

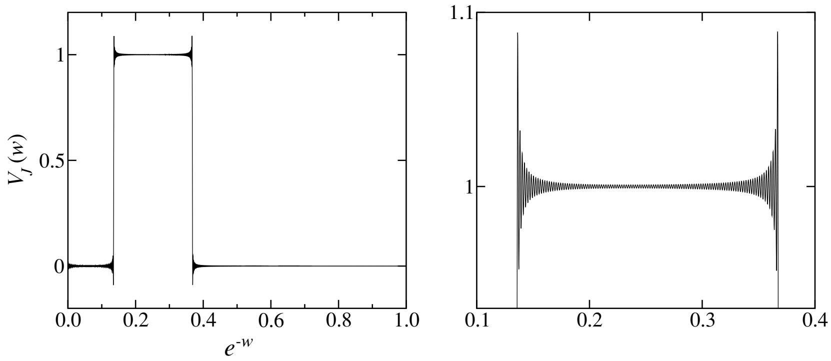

Figure 1. The left panel shows a reconstruction of the square step $V_J(|w|) = 1, 1 < |w| < 2$, $V_J = 0$ otherwise via the M_4^- discrete $J_2(j_i\,\mathrm{e}^{-b|w|})$ mode basis, with the parameter b being set equal to one. The right panel shows a blow-up of the region near the top of the step.

the sum[4]. As we see, the basis is indeed capable of generating the square step to very high accuracy, with its completeness thus being confirmed.

With regard to the plot in figure 1, as can be seen from the blow-up of the region near the top of the step, the mode sum expressly displays the Gibbs phenomenon associated with trying to fit a discontinuity with a complete basis, with there being an overshoot (to near $V_J = 1.1$ in the figure) at the top of the discontinuity and an accompanying undershoot at the bottom, an overshoot and undershoot which as required of the Gibbs phenomenon were explicitly found to get narrower (in $|w|$) as the number of modes in the sum was increased, but not to shorten in height, always reaching close to $V_J = 1.1$ in the figure. We regard the recovering of the Gibbs phenomenon as a very good indicator of the reliability of our construction, and together with the quality of the overall fit itself, as providing very good evidence for completeness of the convergent M_4^- mode basis.

5. Completeness test for divergent M_4^- modes

To test for completeness of the divergent $Y_2(y_i\,\mathrm{e}^{-b|w|})$ plus $\mathrm{e}^{2b|w|}$ mode basis, we try to reconstruct the square step via the expansion

$$V_Y(|w|) = \sum_n V_n Y_2(n\,\mathrm{e}^{-b|w|}/b) + V_0\,\mathrm{e}^{2b|w|}. \tag{35}$$

(In equation (35) we use n to denote the y_i zeros of $Y_1(y)$, and shall use m to denote the j_i zeros of $J_1(y)$.) Now while such a reconstruction might at first be thought unlikely to succeed since every term on the right-hand side of equation (35) diverges badly in the large $|w|$ region where we need the summation to vanish, the various terms in equation (35) are not diverging arbitrarily but, as can be seen from equation (28), are actually all diverging in exactly the same $\mathrm{e}^{2b|w|}$ manner. In consequence of this, we are therefore able to adjust the various coefficients in equation (35) so as to expressly cancel out the divergent part. However, in order to get $V_Y(|w|)$ to actually vanish rather than merely not diverge outside the step, we will also need

[4] This particular completeness test was carried out in collaboration with Dr A H Guth, Dr D I Kaiser and Dr A Nayeri, and grew out of a study of brane-world fluctuations in which they were engaged with one of us (PDM).

to cancel the finite part there as well. Thus, with each $Y_2(y)$ having a leading behaviour of the form $-4/\pi y^2 - 1/\pi$ at small argument, i.e. with equation (35) behaving as

$$V_Y(|w|) \to \mathrm{e}^{2b|w|}\left[V_0 - \frac{4b^2}{\pi}\sum_n \frac{V_n}{n^2}\right] - \frac{1}{\pi}\sum_n V_n \tag{36}$$

at large $|w|$, we need to impose the two conditions

$$\frac{4b^2}{\pi}\sum_n \frac{V_n}{n^2} = V_0, \qquad \sum_n V_n = 0 \tag{37}$$

on the coefficients, with the two leading large $|w|$ terms then being cancelled.

Having thus taken care of the leading behaviour at large $|w|$, we now try to proceed as with our analysis of the expansion of $V_J(|w|)$ in convergent modes. However, we cannot simply apply $\int_0^\infty \mathrm{d}|w|\,\mathrm{e}^{-2b|w|}Y_2(n\,\mathrm{e}^{-b|w|}/b)$ to equation (35) as every overlap integral would diverge. However, we have found it very convenient to apply $\int_0^\infty \mathrm{d}|w|\,\mathrm{e}^{-2b|w|}J_2(m\,\mathrm{e}^{-b|w|}/b)$ to equation (35) instead, where we take m/b to be the j_i zeros of $J_1(y)$. With none of the $J_1(m/b)$ zeros coinciding with any of the zeros of $Y_1(n/b)$,[5] the needed overlap integrals are given (on setting $x = \mathrm{e}^{-b|w|}/b$) by

$$\begin{aligned}\int_0^\infty \mathrm{d}|w|\,\mathrm{e}^{-2b|w|}J_2(m\,\mathrm{e}^{-b|w|}/b)Y_2(n\,\mathrm{e}^{-b|w|}/b) &= b\int_0^{1/b}\mathrm{d}x\,x\,J_2(mx)Y_2(nx)\\ &= bx\left[\frac{nY_1(nx)J_2(mx) - mJ_1(mx)Y_2(nx)}{(m^2-n^2)}\right]\Bigg|_0^{1/b} = \frac{2bm^2}{\pi n^2(n^2-m^2)},\end{aligned} \tag{38}$$

and

$$\begin{aligned}\int_0^\infty \mathrm{d}|w|\,\mathrm{e}^{-2b|w|}J_2(m\,\mathrm{e}^{-b|w|}/b)\,\mathrm{e}^{2b|w|} &= \frac{1}{b}\int_0^{1/b}\frac{\mathrm{d}x}{x}J_2(mx)\\ &= -\frac{1}{b}\int_0^{1/b}\mathrm{d}x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{J_1(mx)}{mx}\right) = \frac{1}{2b},\end{aligned} \tag{39}$$

overlap integrals which despite the badly divergent behaviour of $Y_2(y)$ and $\mathrm{e}^{2b|w|}$ are nonetheless actually finite due to the compensating convergent behaviour of $J_2(y)$. On thus applying $\int_0^\infty \mathrm{d}|w|\,\mathrm{e}^{-2b|w|}J_2(m\,\mathrm{e}^{-b|w|}/b)$ to equation (35), we find that for the square step $V_Y(|w|) = \hat{V}, \alpha \leqslant \mathrm{e}^{-b|w|}/b \leqslant \beta$, $V_Y(|w|) = 0$ otherwise, the expansion coefficients must thus obey

$$\begin{aligned}\frac{V_0}{2b} + \frac{2b}{\pi}\sum_n V_n\frac{m^2}{n^2(n^2-m^2)} &= \frac{V_0}{2b} + \frac{2b}{\pi}\sum_n V_n\left[\frac{1}{(n^2-m^2)} - \frac{1}{n^2}\right]\\ &= \frac{2b}{\pi}\sum_n\frac{V_n}{(n^2-m^2)} = B_m\end{aligned} \tag{40}$$

for all m, where the B_m coefficients are given by

$$\begin{aligned}B_m = -b\hat{V}\int_\alpha^\beta \mathrm{d}x\,x\,J_2(mx) &= \frac{b\hat{V}}{m^2}[2J_0(mx) + mxJ_1(mx)]|_\alpha^\beta\\ &= \frac{b\hat{V}}{m^2}[2J_0(m\beta) + m\beta J_1(m\beta)] - \frac{b\hat{V}}{m^2}[2J_0(m\alpha) + m\alpha J_1(m\alpha)].\end{aligned} \tag{41}$$

[5] The zeros of $J_1(y)$ and $Y_1(y)$ are simple, discrete ones which interlace each other, with first three positive zeros of $J_1(y)$ for instance occurring at 3.832, 7.016 and 10.173, and with the nth positive zero being well approximated by $j_n \approx (n + 1/4)\pi$ when n is large; while the first three positive zeros of $Y_1(y)$ occur at 2.197, 5.430 and 8.596, with the nth positive zero being well approximated by $y_n \approx (n - 1/4)\pi$ when n is large. While these particular approximations do not hold at small n, the parameter n which appears in the $j_n \approx (n + 1/4)\pi$ and $y_n \approx (n - 1/4)\pi$ expressions does denote the number of the zero (counting the first positive zero as $n = 1$), so that for n large or small these expressions give a correct counting of the number of zeros.

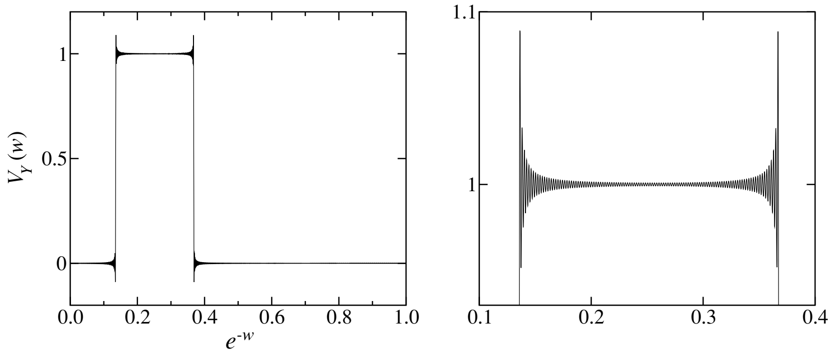

Figure 2. The left panel shows a reconstruction of the square step $V_Y(|w|) = 1, 1 < |w| < 2$, $V_Y = 0$ otherwise via the M_4^- discrete $Y_2(y_i\,\mathrm{e}^{-b|w|})$ plus $\mathrm{e}^{2b|w|}$ mode basis, with the parameter b being set equal to one. The right panel shows a blow-up of the region near the top of the step.

With B_m being given in closed form, equation (40) is thus a set of N equations for N unknowns and can be viewed as an eigenvalue equation for V_n. (While the $J_2(m\,\mathrm{e}^{-b|w|}/b)Y_2(n\mathrm{e}^{-b|w|}/b)$ overlap integrals of equation (38) are finite, the $J_2(m\,\mathrm{e}^{-b|w|}/b)$ and $Y_2(n\,\mathrm{e}^{-b|w|}/b)$ modes are not orthogonal, with equation (40), unlike equation (33), thus not being diagonal in its indices.) The V_n coefficients can thus be determined and, on being found to be finite and rapidly oscillating in sign, lead, for the case of the first 1000 modes in the basis, to the plot displayed in figure 2 (i.e., we restrict to the first 1000 y_i and the first 1000 j_i in equation (40)). As figure 2 thus indicates, and quite spectacularly so, the divergent mode basis is every bit as capable of reconstructing the square step as the convergent one and every bit as capable of recovering the Gibbs phenomenon, and is thus every bit as complete[6]. It is thus invalid to use normalizability as a criterion for discarding modes as non-normalizable modes are fully capable of serving as a complete basis for constructing localized packets[7]. As a final comment, we recall that for the harmonic oscillator wave equation there are two sets of solutions, the sines and the cosines, and both sets are complete. It is hence perfectly reasonable to expect other second-order wave equations to also have two complete sets of bases even if one of them consists entirely of divergent modes.

6. Completeness tests for the anti-de Sitter brane cases

6.1. The basis modes

For AdS_4^+ brane world with warp factor $\mathrm{e}^{A(|w|)} = H\cosh(\sigma - b|w|)/b$ where $\cosh\sigma = b/H$, the transformation $y = \tanh(b|w| - \sigma)$ brings equation (13) to the form

$$\left[(1-y^2)\frac{\mathrm{d}^2}{\mathrm{d}y^2} - 2y\frac{\mathrm{d}}{\mathrm{d}y} + \nu(\nu+1) - \frac{4}{(1-y^2)}\right] f_m(y) = 0. \tag{42}$$

where we have introduced the convenient parameter ν defined by

$$\nu = \left(\frac{9}{4} + \frac{m^2}{H^2}\right)^{1/2} - \frac{1}{2}, \qquad \frac{m^2}{H^2} = (\nu-1)(\nu+2). \tag{43}$$

[6] The reconstruction of the square step using the divergent mode basis is so good that the only perceptible difference between figures 1 and 2 is that in the region close to $\mathrm{e}^{-w} = 0$, the $J_2(m\,\mathrm{e}^{-b|w|}/b)$ contribution is ever so slightly thicker. (The constraints of equation (37) force a more rapid convergence on the $Y_2(n\,\mathrm{e}^{-b|w|}/b)$ mode sum.)

[7] For the M_4^- brane world this is just as well, since it could otherwise not contain any massless graviton.

Equation (42) is recognized as an associated Legendre equation, with its solutions being the associate Legendre functions of the first and second kinds, so that for $m \neq 0$ (namely $\nu \neq 1$) we can set

$$f_m(y) = \alpha_m P_\nu^2(y) + \beta_m Q_\nu^2(y). \tag{44}$$

This solution also applies to one of the $m = 0$ solutions as well, namely $Q_1^2(y)$, a quantity which can be written in terms of the warp factor as $Q_1^2(y) = 2/(1-y^2) = 2\cosh^2(b|w|-\sigma) = 2b^2\,\mathrm{e}^{2A(|w|)}/H^2$, but misses one other solution since $P_1^2(y)$ is kinematically zero. This second $m = 0$ solution can be found by setting $\nu = 1$ in equation (42) and solving it directly, to yield

$$f_0(y) = \alpha_0\left(\frac{2}{(1+y)} - y\right) + \beta_0 Q_1^2(y). \tag{45}$$

Requiring the modes to also obey the junction condition of equation (14) then restricts them according to

$$\alpha_m P_\nu^1(-\tanh\sigma) + \beta_m Q_\nu^1(-\tanh\sigma) = 0, \qquad \alpha_0 = 0, \tag{46}$$

to thus define the AdS_4^+ brane world basis modes.

As functions, all of the functions $P_\nu^1(y)$, $P_\nu^2(y)$, $Q_\nu^1(y)$ and $Q_\nu^2(y)$ possess a cut in the complex y plane which can be located to run from $y = -\infty$ to $y = 1$. For the AdS_4^+ brane world the parameter $y = \tanh(b|w| - \sigma)$ lies in the range $-\tanh\sigma \leqslant y \leqslant 1$, and so in this range the Legendre functions have to be evaluated on the cut (as the real $P_\nu^\mu(y) = (1/2)[\mathrm{e}^{\mathrm{i}\pi\mu/2}P_\nu^\mu(y+\mathrm{i}\epsilon) + \mathrm{e}^{-\mathrm{i}\pi\mu/2}P_\nu^\mu(y-\mathrm{i}\epsilon)]$, $Q_\nu^\mu(y) = (\mathrm{e}^{-\mathrm{i}\pi\mu}/2)[\mathrm{e}^{-\mathrm{i}\pi\mu/2}Q_\nu^\mu(y+\mathrm{i}\epsilon) + \mathrm{e}^{\mathrm{i}\pi\mu/2}Q_\nu^\mu(y-\mathrm{i}\epsilon)]$) where they can then be power series expandable via their relation to hypergeometric functions to yield

$$\begin{aligned}
P_\nu^m(y) &= \frac{(-1)^m\Gamma(\nu+m+1)}{2^m m!\,\Gamma(\nu-m+1)}(1-y^2)^{m/2}F(\nu+m+1, -\nu+m; m+1; (1-y)/2)\\
&= \frac{(-1)^m\Gamma(\nu+m+1)}{2^m m!\,\Gamma(\nu-m+1)}(1-y^2)^{m/2}\left[1 + \frac{(\nu+m+1)(-\nu+m)}{(m+1)1!}\frac{(1-y)}{2}\right.\\
&\quad\left. + \frac{(\nu+m+1)(\nu+m+2)(-\nu+m)(-\nu+m+1)}{(m+1)(m+2)2!}\frac{(1-y)^2}{2^2} + \cdots\right],\\
Q_\nu^m(y) &= \frac{\mathrm{e}^{\mathrm{i}m\pi}2^\nu\Gamma(\nu+1)\Gamma(\nu+m+1)}{\Gamma(2\nu+2)(1+y)^{\nu+1-m/2}(1-y)^{m/2}}F(\nu-m+1, \nu+1; 2\nu+2; 2/(1+y))\\
&= \frac{\mathrm{e}^{\mathrm{i}m\pi}2^\nu\Gamma(\nu+1)\Gamma(\nu+m+1)}{(1+y)^{\nu+1-m/2}(1-y)^{m/2}}\left[\frac{\Gamma(m)}{\Gamma(\nu+1)\Gamma(\nu+m+1)}\right.\\
&\quad\times\sum_{n=0}^{m-1}\frac{(\nu-m+1)_n(\nu+1)_n}{(1-m)_n n!}\frac{(y-1)^n}{(y+1)^n} + \frac{(-1)^m(y-1)^m}{\Gamma(\nu-m+1)\Gamma(\nu+1)(y+1)^m}\\
&\quad\times\sum_{n=0}^{\infty}\frac{(\nu+1)_n(\nu+m+1)_n}{(n+m)!n!}\frac{(y-1)^n}{(y+1)^n}\Big[\psi(n+1)+\psi(n+m+1)\\
&\quad\left.\left. - \psi(\nu+1+n) - \psi(\nu+m+1+n) - \log\left(\frac{1-y}{1+y}\right)\right]\right]
\end{aligned} \tag{47}$$

when μ is a general positive integer m. (In equation (47), $\psi(y)$ denotes $(\mathrm{d}\Gamma(y)/\mathrm{d}y)/\Gamma(y)$ and $(a)_n$ denotes $\Gamma(a+n)/\Gamma(a)$.) From equation (47) we see that in the $-\tanh\sigma \leqslant y \leqslant 1$ range of interest the $P_\nu^2(y)$ functions are well behaved, behaving as y approaches one from below (namely as $|w| \to \infty$) as

$$P_\nu^2(y \to 1) \to P(\nu)\left[(1-y) - \frac{(1-y)^2(\nu^2+\nu-3)}{6}\right] \tag{48}$$

where

$$P(\nu) = \frac{\nu(\nu^2-1)(\nu+2)}{4}, \tag{49}$$

to thus be fully normalizable and have finite normalization

$$N_\nu = \int_{-\infty}^{\infty} \mathrm{d}w\, \mathrm{e}^{-2A}\big[P_\nu^2(|w|)\big]^2 = 2\int_0^\infty \mathrm{d}|w|\,\mathrm{e}^{-2A}\big[P_\nu^2(|w|)\big]^2 = \frac{2b}{H^2}\int_{-\tanh\sigma}^{1}\mathrm{d}y\big[P_\nu^2(y)\big]^2. \tag{50}$$

However, unlike the $P_\nu^2(y)$, the $Q_\nu^2(y)$ are all found to diverge at $y = 1$, behaving there as

$$Q_\nu^2(y\to 1) \to \frac{1}{(1-y)} + \frac{(\nu^2+\nu-1)}{2} + O((1-y)\ln(1-y)), \tag{51}$$

and thus in the AdS_4^+ brane world none of $Q_\nu^2(y)$, and particularly the massless $Q_1^2(y)$ graviton, are normalizable. We shall thus seek to construct complete bases in both the normalizable and non-normalizable sectors.

6.2. Completeness test for convergent AdS_4^+ *modes*

To construct a complete basis out of normalizable modes alone requires that the normalizable $P_\nu^2(y)$ satisfy equation (46) all on their own, with the eigenmodes then needing to satisfy

$$P_\nu^1(-\tanh\sigma) = P_\nu^1(-(1-H^2/b^2)^{1/2}) = 0. \tag{52}$$

For arbitrary σ the solutions to equation (52) cannot be written in a closed form, but on noting that for one particular value of σ, namely $\sigma = 0$ (i.e., $H = b$), $P_\nu^1(0)$ is known in closed form as

$$P_\nu^1(0) = \frac{2\pi^{1/2}}{\Gamma(\nu/2+1/2)\Gamma(-\nu/2)}, \tag{53}$$

to thus be zero at $\nu = 2, 4, 6, \ldots$, we see that on solving for an arbitrary given σ numerically an infinite discrete set of allowed ν values will then be found to ensue[8]. The normalizable mode sector of AdS_4^+ is thus discrete and infinite, a result first obtained in [7] by directly numerically solving equation (13).

To test for completeness of the normalizable AdS_4^+ mode basis, we need to find a set of coefficients V_m for which the expansion

$$V_P = \sum_m V_m P_\nu^2(y) \tag{54}$$

reproduces the square step $V_P = \hat{V}$ when $|w_1| < |w| < |w_2|$, $V_P = 0$ otherwise. With the $P_\nu^2(y)$ modes being orthogonal, the coefficients are readily given as $V_m = B_m/N_\nu$ where N_ν is the normalization factor given in equation (50), where m and ν are related as in equation (43), and where some standard properties of the associated Legendre functions allow the B_m coefficients to be written as

$$\begin{aligned} B_m &= \hat{V}\int_{|w_1|}^{|w_2|}\mathrm{d}|w|\,\mathrm{e}^{-2A}P_\nu^2(|w|) = \frac{\hat{V}b}{H^2}\int_{y_1}^{y_2}\mathrm{d}y\,P_\nu^2(y) = \frac{\hat{V}b}{H^2}\int_{y_1}^{y_2}\mathrm{d}y(1-y^2)\frac{\mathrm{d}^2P_\nu(y)}{\mathrm{d}y^2} \\ &= \frac{\hat{V}b}{H^2}\int_{y_1}^{y_2}\mathrm{d}y\left[\frac{\mathrm{d}}{\mathrm{d}y}[(2-\nu)yP_\nu + \nu P_{\nu-1}] - 2P_\nu\right] \end{aligned}$$

[8] The typical case of $\tanh\sigma = 0.9$ (namely $H/b = 0.436$) yields $\nu = 1.088, 2.216$ and 3.362 as the three lowest positive solutions to $P_\nu^1(-\tanh\sigma) = 0$, with the nth positive zero being well approximated by $\nu_n \approx (n+1/4)\pi/\arccos(-\tanh\sigma) - 1/2$ when n is large.

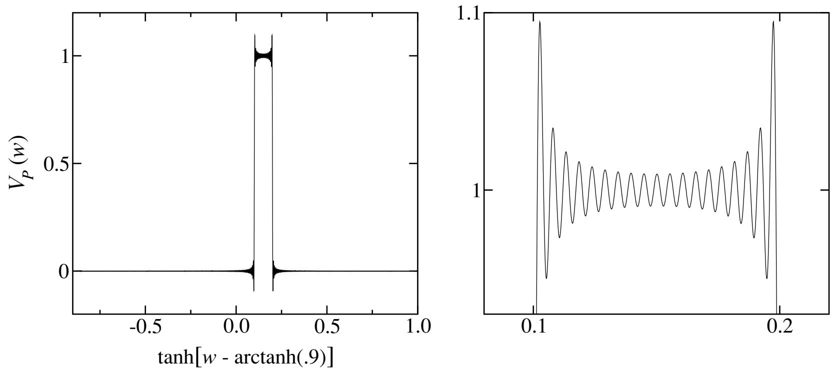

Figure 3. The left panel shows a reconstruction of the square step $V_P(w) = 1, 0.1 \leqslant \tanh(b|w| - \mathrm{arctanh}(0.9)) \leqslant 0.2$, $V_P(w) = 0$ otherwise, via the AdS_4^+ discrete $P_\nu^2(\tanh(b|w| - \sigma))$ mode basis in the typical case where $\tanh\sigma = 0.9$, $H/b = 0.436$ and $b = 1$. The right panel shows a blow-up of the region near the top of the step.

$$
\begin{aligned}
&= \frac{\hat{V} b}{H^2} \int_{y_1}^{y_2} \mathrm{d}y \frac{\mathrm{d}}{\mathrm{d}y} \left[\frac{(2-\nu)}{(2\nu+1)} [(\nu+1) P_{\nu+1} + \nu P_{\nu-1}] + \nu P_{\nu-1} - \frac{2}{(2\nu+1)} (P_{\nu+1} - P_{\nu-1}) \right] \\
&= \frac{\hat{V} b}{H^2} \left[\frac{(\nu+1)(\nu+2) P_{\nu-1} - \nu(\nu-1) P_{\nu+1}}{2\nu+1} \right] \Bigg|_{y_1}^{y_2}.
\end{aligned} \tag{55}
$$

With every quantity which appears in equation (54) now being known, $V_P(|w|)$ can readily be plotted, and we display it in figure 3 as evaluated through the use of the first 1000 modes in the sum. As we see, the basis is indeed capable of generating the square step to very high accuracy, and with it expressly displaying the Gibbs phenomenon[9], its completeness is thus confirmed.

6.3. *Completeness test for divergent* AdS_4^+ *modes*

With the massless AdS_4^+ graviton with divergent warp factor wavefunction $f_0(y) = \beta_0 Q_1^2(y) = 2\beta_0/(1 - y^2)$ obeying the junction condition, it could also belong to a complete basis of divergent $Q_\nu^2(y)$ modes (modes which according to equation (51) actually diverge in precisely the same $1/(1 - y)$ way near $y = 1$ as the massless graviton itself) if the $Q_\nu^2(y)$ modes were to satisfy the junction condition on their own, i.e. if they were to obey

$$
Q_\nu^1(-\tanh\sigma) = Q_\nu^1(-(1 - H^2/b^2)^{1/2}) = 0. \tag{56}
$$

With equation (56) being found to possess an infinite set of discrete solutions for the arbitrary σ,[10] we shall thus seek to expand the localized square step $V_Q = \hat{V}$ when $|w_1| \leqslant |w| \leqslant |w_2|$, $V_Q = 0$ otherwise, in terms of these solutions as

$$
V_Q = \sum_n V_n Q_\nu^2(y) + \frac{V_0}{1 - y^2}. \tag{57}
$$

[9] It is possible that this might perhaps be the first time that the Gibbs phenomenon has explicitly been demonstrated for associated Legendre functions, and especially for the divergent $Q_\nu^2(y)$ modes which we show below.

[10] The typical case of $\tanh\sigma = 0.9$ yields $\nu = 0.536, 1.649$ and 2.788 as the three lowest positive solutions to $Q_\nu^1(-\tanh\sigma) = 0$, with the nth positive zero being well approximated by $\nu_n \approx (n - 1/4)\pi/\arccos(-\tanh\sigma) - 1/2$ when n is large, with the zeros of $P_\nu^1(-\tanh\sigma) = 0$ and $Q_\nu^1(-\tanh\sigma) = 0$ thus interlacing each other. As regards the $Q_\nu^1(-\tanh\sigma) = 0$ solutions, we note additionally that the lowest positive one actually corresponds to an $m^2 < 0$ tachyon since it has $\nu < 1$.

(For clarity we use n^2 here to denote the squared masses $n^2/H^2 = (\nu - 1)(\nu + 2)$ of the $Q^2_\nu(y)$ sector modes, and use m^2 for the $P^2_\nu(y)$ sector.) Given the asymptotic limit exhibited in equation (51), in order to first cancel both the leading $1/(1-y)$ term and the next to leading $O(1)$ term from the right-hand side of equation (57), we must constrain the V_n coefficients according to

$$\sum_n V_n + \frac{V_0}{2} = 0, \qquad \frac{1}{2}\sum_n V_n(\nu^2 + \nu - 1) + \frac{V_0}{4} = 0, \tag{58}$$

to thus enable us to reexpress the square step expansion as

$$V_Q = \sum_n V_n \left[Q^2_\nu(y) - \frac{2}{1-y^2} \right], \tag{59}$$

as subject to the constraint

$$\sum_n V_n[\nu^2 + \nu - 2] = \sum_n V_n \frac{n^2}{H^2} = 0. \tag{60}$$

While we cannot apply $\int_0^\infty \mathrm{d}|w|\, \mathrm{e}^{-2A} Q^2_\nu(y)$ to equation (59) as every overlap integral would diverge, finite overlap integrals are obtained if we instead apply $\int_0^\infty \mathrm{d}|w|\, \mathrm{e}^{-2A} P^2_{\nu'}(y)$, where we use ν' to label the $P^2_\nu(y)$ sector so that its squared masses are given by $m^2/H^2 = (\nu' - 1)(\nu' + 2)$. With none of the $P^1_{\nu'}(-\tanh\sigma)$ and $Q^1_\nu(-\tanh\sigma)$ zeros being found to coincide, via equations (16), (48) and (51), the needed overlap integrals are found to be of the form

$$\int_0^\infty \mathrm{d}|w|\, \mathrm{e}^{-2A} P^2_{\nu'}(y) Q^2_\nu(y) = \frac{4bP(\nu')}{(m^2 - n^2)} \tag{61}$$

($P(\nu')$ is given in equation (49)), and are indeed finite, just as required. With the overlap integral which involves the massless graviton mode being given by

$$\int_0^\infty \mathrm{d}|w|\, \mathrm{e}^{-2A} \frac{P^2_{\nu'}(y)}{(1-y^2)} = \frac{2bP(\nu')}{m^2}, \tag{62}$$

the application of $\int_0^\infty \mathrm{d}|w|\, \mathrm{e}^{-2A} P^2_{\nu'}(y)$ to equation (59) thus yields

$$4b \sum_n V_n P(\nu') \left[\frac{1}{(m^2 - n^2)} - \frac{1}{m^2} \right] = \sum_n V_n \frac{b(m^2 + 2H^2)n^2}{H^4(m^2 - n^2)} = B_m, \tag{63}$$

where B_m is the same function that was already given earlier in equation (55).

Given equation (63), the V_n coefficients can now be found numerically, and lead, for the case of the first 1000 modes in the basis, to the plot displayed in figure 4 (i.e., we restrict to the first 1000 $P^1_{\nu'}(-\tanh\sigma)$ zeros and the first 1000 $Q^1_\nu(-\tanh\sigma)$ zeros). As figure 4 thus indicates, the divergent mode basis is every bit as capable of reconstructing the square step as the convergent one and every bit as capable of recovering the Gibbs phenomenon, and is thus every bit as complete[11]. Once again then we see that it is invalid to use normalizability as a criterion for discarding modes, and in this regard we differ from the view of [7] that it is permissible to discard modes such as the massless AdS_4^+ graviton simply because they are not normalizable[12].

[11] The construction is so good that the only perceptible difference between figures 3 and 4 is that in the regions close to the edges of the steps the Gibbs phenomenon overshoot, as shown in figure 3 blow-up is ever so slightly closer to 1.1 than the one shown in the blow-up of figure 4.

[12] Since the negative tension AdS_4^- brane world with divergent warp factor $\mathrm{e}^{A(|w|)} = H\cosh(\sigma + b|w|)/b$ also has convergent $P^2_\nu(y)$ and divergent $Q^2_\nu(y)$ modes (where now $y = \tanh(b|w| + \sigma)$ with range $\tanh\sigma \leqslant y \leqslant 1$), its structure is analogous to that of the divergent warp factor AdS_4^+ world, and so we do not seek completeness tests for it here.

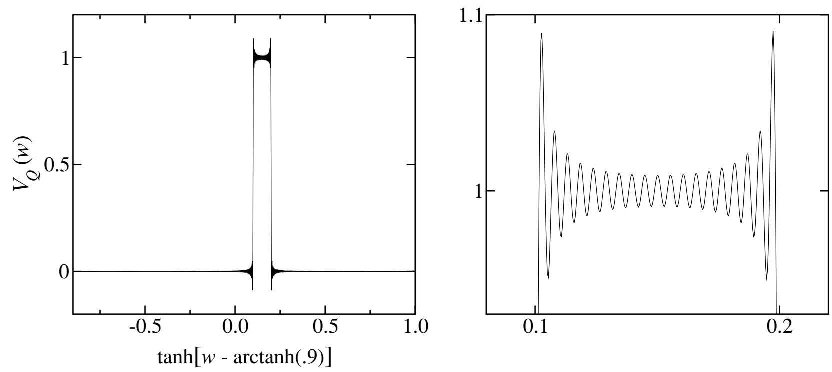

Figure 4. The left panel shows a reconstruction of the square step $V_Q(w) = 1, 0.1 \leqslant \tanh(b|w| - \text{arctanh}(0.9)) \leqslant 0.2$, $V_Q(w) = 0$ otherwise, via the AdS_4^+ discrete $Q_\nu^2(\tanh(b|w| - \sigma))$ plus $\cosh^2(\tanh(b|w| - \sigma))$ mode basis in the typical case where $\tanh\sigma = 0.9$, $H/b = 0.436$ and $b = 1$. The right panel shows a blow-up of the region near the top of the step.

7. Completeness tests for the de Sitter brane cases

7.1. The basis modes

For $\text{dS}_4^\pm$ brane worlds with warp factor $e^{A(|w|)} = H\sinh(\sigma \mp b|w|)/b$ where $\sinh\sigma = b/H$, the transformation $y = \coth(\sigma \mp b|w|)$ brings equation (13) to the form

$$\left[(1 - y^2)\frac{d^2}{dy^2} - 2y\frac{d}{dy} + \nu(\nu + 1) - \frac{4}{(1 - y^2)}\right] f_m(y) = 0, \tag{64}$$

where we have introduced the convenient parameter ν defined by

$$\nu = \left(\frac{9}{4} - \frac{m^2}{H^2}\right)^{1/2} - \frac{1}{2}, \qquad \frac{m^2}{H^2} = (1 - \nu)(\nu + 2). \tag{65}$$

Recognizing equation (64) to be the previously discussed associated Legendre equation, its $m \neq 0$ (namely $\nu \neq 1$) solutions are given as

$$f_m(y) = \alpha_m P_\nu^2(y) + \beta_m Q_\nu^2(y), \tag{66}$$

while its $\nu = 1$ solutions are of the form

$$f_0(y) = \alpha_0\left(\frac{2}{(1 + y)} - y\right) + \beta_0 Q_1^2(y). \tag{67}$$

Requiring the modes to also obey the junction condition of equation (14) then restricts them according to

$$\alpha_m P_\nu^1(\coth\sigma) + \beta_m Q_\nu^1(\coth\sigma) = 0, \qquad \alpha_0 = 0, \tag{68}$$

to thus define the $\text{dS}_4^\pm$ brane-world basis modes.

While the dS_4^+ and dS_4^- basis modes are quite similar to each other in their generic structure, they differ from each other significantly in one crucial regard. Specifically, unlike the dS_4^- warp factor $e^{A(|w|)} = H\sinh(\sigma + b|w|)/b$ which never vanishes (σ having been defined to be positive), the dS_4^+ warp factor $e^{A(|w|)} = H\sinh(\sigma - b|w|)/b$ has a zero at $b|w| = \sigma$. With a null signal taking an infinite amount of time to travel from the brane to the location of this zero, this zero serves as a horizon for an observer on the brane [8], with the brane observer only being sensitive to fluctuation modes in the $\sigma \geqslant b|w| \geqslant 0$ region. With the dS_4^+ parameter $y = \coth(\sigma - b|w|)$ lying in the range $\coth\sigma \leqslant y \leqslant \infty$, we see that y is infinite at the dS_4^+ horizon. Then, with the associated Legendre functions behaving as $P_\nu^2(y) \to O(y^\nu) + O(y^{-\nu-1})$, $Q_\nu^2(y) \to O(y^{-\nu-1})$ as $y \to \infty$, the $\nu = 1$ massless dS_4^+ graviton and all dS_4^+ modes with complex $\nu = -1/2 \pm \mathrm{i}\,(m^2/H^2 - 9/4)^{1/2}$ will be normalizable within the horizon[13]. With the massless graviton and a massive continuum of modes with $m^2/H^2 \geqslant 9/4$ which satisfy the junction condition of equation (68) by an interplay (of the real $P_\nu^2(y)$ and the real part of $Q_\nu^2(y)$) thus providing a conventional continuum normalized complete basis in the sense of equations (1)–(5), as with the M_4^+ brane world, in the dS_4^+ brane world there is no need to test explicitly for completeness.

However, for dS_4^- the situation is quite different since there is now no vanishing of the warp factor and no horizon, with the coordinate $|w|$ now extending all the way to infinity, and with the parameter $y = \coth(\sigma + b|w|)$ instead now lying in the $1 \leqslant y \leqslant \coth\sigma = (1 + H^2/b^2)^{1/2}$ range. Unlike the previously discussed AdS_4^+ brane world case where y approached one from below as $|w|$ went to infinity, in the dS_4^- case y instead approaches one from above in the large $|w|$ limit, with equations (48) and (51) having to be replaced by the limits

$$\begin{aligned} P_\nu^2(y \to 1) &\to P(\nu)\left[(y-1) + \frac{(y-1)^2(\nu^2+\nu-3)}{6}\right] \\ Q_\nu^2(y \to 1) &\to \frac{1}{(y-1)} - \frac{(\nu^2+\nu-1)}{2} + O\left((y-1)\ln(y-1)\right), \end{aligned} \tag{69}$$

where $P(\nu) = \nu(\nu^2 - 1)(\nu + 2)/4$ is as given in equation (49). Since the $P_\nu^2(y)$ are well behaved at $y = 1$, while $Q_\nu^2(y)$ diverge there, as with the AdS_4^+ case, the normalizable sector will consist of all $P_\nu^2(\coth(\sigma + b|w|))$ modes which satisfy the junction condition on their own according to

$$P_\nu^1(\coth\sigma) = P_\nu^1((1 + H^2/b^2)^{1/2}) = 0, \tag{70}$$

while the non-normalizable sector will consist of the divergent warp factor wavefunction $Q_1^2(\coth(\sigma + b|w|))$ $(=2/(y^2 - 1)$ in $y > 1)$ massless graviton and all massive

[13] With the arbitrary hypergeometric function $F(a, b; c, z)$ being equal to one when its argument z is taken to be zero, the large y limits of $P_\nu^2(y)$ and $Q_\nu^2(y)$ are readily obtained from their $|y| > 1$ hypergeometric function representations of the form $P_\nu^\mu(y) = 2^{\nu+1}\Gamma(-2\nu-1)\Gamma^{-1}(-\nu)\Gamma^{-1}(-\nu-\mu)(y+1)^{\mu/2-\nu-1}(y-1)^{-\mu/2}F(\nu+1, \nu-\mu+1; 2\nu+2, 2/(1+y)) + 2^{-\nu}\Gamma(2\nu+1)\Gamma^{-1}(\nu+1)\Gamma^{-1}(\nu-\mu+1)(y+1)^{\mu/2+\nu}(y-1)^{-\mu/2}F(-\nu, -\nu-\mu; -2\nu, 2/(1+y))$, $Q_\nu^\mu(y) = \mathrm{e}^{\mathrm{i}\mu\pi}2^{-\nu-1}\pi^{1/2}\Gamma(\nu+\mu+1)\Gamma^{-1}(\nu+3/2)y^{-\nu-\mu-1}(y^2-1)^{\mu/2}F(\nu/2+\mu/2+1, \nu/2+\mu/2+1/2; \nu+3/2, 1/y^2)$. While these representations show that $P_\nu^\mu(y)$ and $Q_\nu^\mu(y)$ will in general be complex in the $|y| > 1$ region, the form for $P_\nu^\mu(y)$ shows that it will actually be real when y and μ are real and the parameter ν takes the value $\nu = -1/2 + \mathrm{i}\lambda$ where λ is real, a value for which the quantity $\nu(\nu+1) = (\nu+1/2)^2 - 1/4$ which appears in the defining equation for the associated Legendre functions of equation (64) is then given as the real $\nu(\nu+1) = -\lambda^2 - 1/4$. With equation (64) remaining real at $\nu = -1/2 + \mathrm{i}\lambda$, for such values of ν the then real $P_\nu^2(y)$ and the real and imaginary parts of $Q_\nu^2(y)$ will all separately obey it. However, since equation (64) can only have two independent solutions, it must be the case that one of these three classes of solutions is redundant. On noting that no matter what the value of ν, the divergent part of $Q_\nu^2(y)$ at $y = 1$ is real while $P_\nu^2(y)$ is well behaved there, we thus anticipate that when y is real and greater than one, it must be the (thus well behaved at $y = 1$) imaginary part of $Q^2_{-1/2+\mathrm{i}\lambda}(y)$ which must coincide with the real $P^2_{-1/2+\mathrm{i}\lambda}(y)$; and since it is not immediately obvious how one may explicitly check such a connection analytically, we have instead confirmed it numerically. In the following, then we can restrict the discussion to the use of $P^2_{-1/2+\mathrm{i}\lambda}(y)$ and $\mathrm{Re}[Q^2_{-1/2+\mathrm{i}\lambda}(y)]$ as basis modes (in both the dS_4^+ and the dS_4^- brane worlds). As well as enabling us to show that $P_\nu^\mu(y)$ is real for real y, real μ and complex $\nu = -1/2 + \mathrm{i}\lambda$, the above representations of the $P_\nu^\mu(y)$ and $Q_\nu^\mu(y)$ are also of use for actual computational purposes when y is greater than one, since for argument $|z| < 1$ a hypergeometric function can be represented as the absolutely convergent power series $F(a, b; c, z) = [\Gamma(c)/\Gamma(a)\Gamma(b)]\sum_{n=0}^{\infty}\Gamma(a+n)\Gamma(b+n)z^n/[\Gamma(c+n)n!]$. Moreover, for large values of the parameter λ, the functions $P^\mu_{-1/2+\mathrm{i}\lambda}(y)$ and $\mathrm{Re}[Q^\mu_{-1/2+\mathrm{i}\lambda}(y)]$ can even be approximated by $P^\mu_{-1/2+\mathrm{i}\lambda}(\cosh\theta) = \lambda^{\mu-1/2}(2/\pi\sinh\theta)^{1/2}\cos(\lambda\theta + \mu\pi/2 - \pi/4) - \lambda^{\mu-3/2}(1/2\pi\sinh\theta)^{1/2}(\mu-1/2)(\mu+1/2)\coth\theta\sin(\lambda\theta + \mu\pi/2 - \pi/4)$ and $\mathrm{Re}[Q^\mu_{-1/2+\mathrm{i}\lambda}(\cosh\theta)] = \lambda^{\mu-1/2}(\pi/2\sinh\theta)^{1/2}\cos(\lambda\theta + \mu\pi/2 + \pi/4) - \lambda^{\mu-3/2}(\pi/8\sinh\theta)^{1/2}(\mu-1/2)(\mu+1/2)\coth\theta\sin(\lambda\theta + \mu\pi/2 + \pi/4)$. (It is necessary to carry the first non-leading terms here since the oscillatory leading terms can vanish at some specific θ values.)

$Q^2_\nu(\coth(\sigma + b|w|))$ modes which obey

$$Q^1_\nu(\coth\sigma) = Q^1_\nu((1 + H^2/b^2)^{1/2}) = 0. \quad (71)$$

While this pattern is thus quite similar to the situation found in the AdS_4^+ case, the dS_4^- brane world differs from it in one key regard, namely that the parameter y is required to be greater or equal to one rather than less than or equal to it, and thus the completeness of its mode bases requires independent testing.

7.2. *Completeness test for convergent* dS_4^- *modes*

With the general equation (16) taking the form

$$\left(\frac{m_1^2}{H^2} - \frac{m_2^2}{H^2}\right)\int_1^{\coth\sigma} \mathrm{d}y\, f_{m_1}(y) f_{m_2}(y)$$
$$= \lim_{y\to 1}\left[(y^2-1) f_{m_2}(y)\frac{\mathrm{d}f_{m_1}(y)}{\mathrm{d}y} - (y^2-1) f_{m_1}(y)\frac{\mathrm{d}f_{m_2}(y)}{\mathrm{d}y}\right] \quad (72)$$

in the dS_4^- case, and with the $P^2_\nu(y)$ modes behaving near $y = 1$ as in equation (69), the $P^2_\nu(y)$ modes form an orthonormal basis, and we can thus normalize thus them according to

$$N_\nu = \int_{-\infty}^{\infty} \mathrm{d}w\, \mathrm{e}^{-2A}\left[P^2_\nu(|w|)\right]^2 = \frac{2b}{H^2}\int_1^{\coth\sigma} \mathrm{d}y\left[P^2_\nu(y)\right]^2. \quad (73)$$

With the $\coth\sigma$ argument of $P^1_\nu(\coth\sigma)$ in equation (70) being greater than one, the $P^1_\nu(\coth\sigma) = 0$ condition has no solutions with real ν. Rather, all of its solutions are of the form $\nu = -1/2 + \mathrm{i}\lambda$ where λ is real and discrete[14]. According to equation (65), for such solutions the associated squared masses obey $m^2/H^2 = 9/4 + \lambda^2$ and are thus nicely positive. Additionally, as noted previously, for the particular choice of $\nu = -1/2 + \mathrm{i}\lambda$, the $P^2_\nu(y)$ mode wavefunctions themselves are real.

Having now explicitly identified the dS_4^- normalizable mode basis, to test for completeness we need to find a set of coefficients V_m for which the expansion

$$\hat{V}_P = \sum_m V_m P^2_\nu(y) \quad (74)$$

reproduces the square step $\hat{V}_P = \hat{V}$ when $|w_1| < |w| < |w_2|$, $\hat{V}_P = 0$ otherwise. With the $P^2_\nu(y)$ modes being orthogonal, the coefficients are readily given as $V_m = B_m/N_\nu$ where N_ν is the normalization factor given in equation (73), where m and ν are related as in equation (65), and where the B_m are given as

$$B_m = \hat{V}\int_{|w_1|}^{|w_2|} \mathrm{d}|w|\, \mathrm{e}^{-2A} P^2_\nu(|w|) = -\frac{\hat{V}b}{H^2}\int_{y_1}^{y_2} \mathrm{d}y\, P^2_\nu(y) = -\frac{\hat{V}b}{H^2}\int_{y_1}^{y_2} \mathrm{d}y (y^2-1)\frac{\mathrm{d}^2 P_\nu(y)}{\mathrm{d}y^2}$$
$$= -\frac{\hat{V}b}{H^2}\int_{y_1}^{y_2} \mathrm{d}y\left[\frac{\mathrm{d}}{\mathrm{d}y}\left[(\nu-2)yP_\nu - \nu P_{\nu-1}\right] + 2P_\nu\right]$$

[14] With the dS_4^- brane world range for y being restricted to the finite range $1 \leqslant y \leqslant \coth\sigma$, in cases in which we restrict to $\coth\sigma < 3$, we are actually able to use an extremely compact representation for evaluation of $P^2_\nu(y)$, $P^1_\nu(y)$ and $P_\nu(y)$, namely the form $P^m_\nu(y) = (y^2-1)^{m/2}\Gamma(\nu+m+1)F(-\nu+m, \nu+m+1, m+1, (1-y)/2)/[2^m m!\Gamma(\nu-m+1))]$ which holds for any positive integer m, and the limiting form $P_\nu(y) = F(-\nu, \nu+1, 1, (1-y)/2))$ which holds when $m = 0$, as each of these hypergeometric function representations can be written as a power series which is absolutely convergent over the entire $1 \leqslant y \leqslant 3$ range. From these representation we find in a typical case with $\coth\sigma = 1.1$ that the three lowest positive λ solutions to $P^1_{-1/2+\mathrm{i}\lambda}(\coth\sigma) = 0$ are given as $\lambda = 8.624$, 15.808 and 22.930, with the nth positive solution being well approximated by $\lambda_n \approx (n + 1/4)\pi/\mathrm{arccosh}(\coth\sigma)$ when n is large.

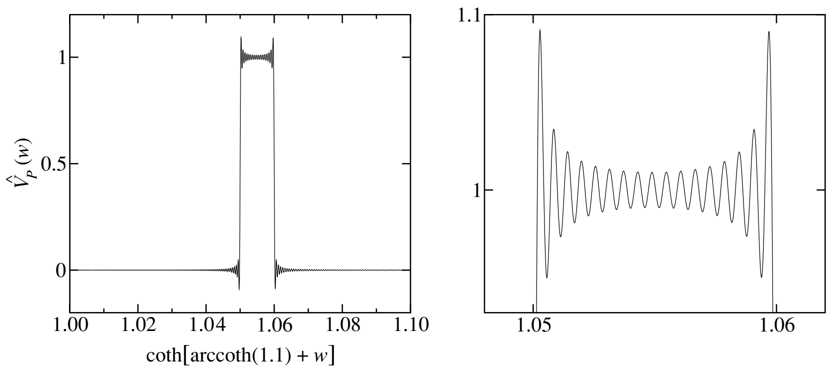

Figure 5. The left panel shows a reconstruction of the square step $\hat{V}_P(w) = 1, 1.05 \leqslant \coth(\mathrm{arccoth}(1.1) + b|w|) \leqslant 1.06$, $\hat{V}_P(w) = 0$ otherwise, via the dS_4^- discrete $P_\nu^2(\coth(\sigma + b|w|))$ mode basis in the typical case where $\coth\sigma = 1.1$, $H/b = 0.458$ and $b = 1$. The right panel shows a blow-up of the region near the top of the step.

$$= -\frac{\hat{V}b}{H^2}\int_{y_1}^{y_2}\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{(\nu-2)}{(2\nu+1)}[(\nu+1)P_{\nu+1} + \nu P_{\nu-1}] - \nu P_{\nu-1} + \frac{2}{(2\nu+1)}(P_{\nu+1} - P_{\nu-1})\right]$$
$$= -\frac{\hat{V}b}{H^2}\left[\frac{[\nu(\nu-1)P_{\nu+1} - (\nu+1)(\nu+2)P_{\nu-1}]}{2\nu+1}\right]\Bigg|_{y_1}^{y_2}. \tag{75}$$

Given equation (75), $\hat{V}_P(|w|)$ can readily be plotted, and we display it in figure 5 as evaluated through the use of the first 500 modes in the sum. As we see, the basis is indeed capable of generating the square step to very high accuracy, and with it also nicely displaying the Gibbs phenomenon, its completeness is thus confirmed.

7.3. Completeness test for divergent dS_4^- *modes*

As with the $P_\nu^1(\coth\sigma) = 0$ condition, the solutions to $Q_\nu^1(\coth\sigma) = 0$ are also all of the form $\nu = -1/2 + \mathrm{i}\lambda$ where λ is again real and discrete, with the solutions to $P_\nu^1(\coth\sigma) = 0$ and $Q_\nu^1(\coth\sigma) = 0$ being found to interlace each other[15]. With it being only the real parts of the $Q_\nu^2(y)$ wavefunctions with $\nu = -1/2 + \mathrm{i}\lambda$ and y real which are independent of the real $P_\nu^2(y)$, the non-normalizable dS_4^- brane world mode basis consists of the massless graviton with its real warp factor wavefunction plus the real parts of the $Q_\nu^2(y)$ wavefunctions with the appropriate $\nu = -1/2 + \mathrm{i}\lambda$. Then, with the $y \to 1$ limit of equation (69) holding for the general $Q_\nu^2(y)$ with arbitrary ν, we see that the real parts of the $Q_\nu^2(y)$ wavefunctions all have the same $1/(y-1)$ leading behaviour at $y = 1$ as the massless graviton itself, with the non-normalizable modes all diverging at $y = 1$ at one and the same rate.

In order to test for completeness in the $\mathrm{Re}\big[Q_\nu^2(y)\big]$ plus massless graviton sector, we need to expand the localized square step $\hat{V}_Q = \hat{V}$ when $|w_1| \leqslant |w| \leqslant |w_2|$, $\hat{V}_Q = 0$ otherwise, in terms of these solutions as

$$\hat{V}_Q = \sum_n V_n \mathrm{Re}\big[Q_\nu^2(y)\big] + \frac{V_0}{y^2 - 1}. \tag{76}$$

[15] The typical case of $\coth\sigma = 1.1$ yields $\lambda = 4.928, 12.231$ and 19.373 as the three lowest positive λ solutions to $\mathrm{Re}[Q_{-1/2+\mathrm{i}\lambda}^1(\coth\sigma)] = 0$, with the nth positive solution being well approximated by $\lambda_n \approx (n - 1/4)\pi/\mathrm{arccosh}(\coth\sigma)$ when n is large.

(As previously, for clarity we use n^2 here to denote the squared masses of the $Q_\nu^2(y)$ sector modes, and use m^2 for the $P_\nu^2(y)$ sector.) Given the asymptotic limit exhibited in equation (69), in order to cancel both the leading $1/(y-1)$ term and the next to leading $O(1)$ term from the right-hand side of equation (76), we must constrain the V_n coefficients according to

$$\sum_n V_n + \frac{V_0}{2} = 0, \qquad \frac{1}{2}\sum_n V_n(\nu^2+\nu-1) + \frac{V_0}{4} = 0, \tag{77}$$

to thus enable us to reexpress the square step expansion as

$$\hat{V}_Q = \sum_n V_n \left[\mathrm{Re}\big[Q_\nu^2(y)\big] - \frac{2}{y^2-1}\right], \tag{78}$$

as subject to the constraint

$$\sum_n V_n[\nu^2+\nu-2] = -\sum_n V_n \frac{n^2}{H^2} = 0. \tag{79}$$

On now applying $\int_0^\infty \mathrm{d}|w|\,\mathrm{e}^{-2A} P_{\nu'}^2(|w|) = (b/H^2)\int_1^{\coth\sigma} \mathrm{d}y\, P_{\nu'}^2(y)$ to equation (78) where $\nu'^2+\nu'-2 = -m^2/H^2$, use of the relations

$$\int_1^{\coth\sigma} \mathrm{d}y\, P_{\nu'}^2(y)\mathrm{Re}\big[Q_\nu^2(y)\big] = \frac{4H^2 P(\nu')}{(m^2-n^2)}, \tag{80}$$

$$\int_1^{\coth\sigma} \mathrm{d}y \frac{P_{\nu'}^2(y)}{(y^2-1)} = \frac{2H^2 P(\nu')}{m^2}, \tag{81}$$

which follow from equations (69) and (72) (with $P(\nu') = \nu'(\nu'^2-1)(\nu'+2)/4$ now being given by $m^2(m^2-2H^2)/4H^4$) then yields

$$4b\sum_n V_n P(\nu')\left[\frac{1}{(m^2-n^2)} - \frac{1}{m^2}\right] = \sum_n V_n \frac{b(m^2-2H^2)n^2}{H^4(m^2-n^2)} = B_m, \tag{82}$$

where B_m is the same function that was already given earlier in equation (75).

Given equation (82), the V_n coefficients can now be found numerically, and lead, for the case of the first 500 modes in the basis, to the plot displayed in figure 6 (i.e., we restrict to the first 500 $P_{\nu'}^1(\coth\sigma)$ zeros and the first 500 $\mathrm{Re}[Q_\nu^1(\coth\sigma)]$ zeros). As figure 6 thus indicates, the divergent mode basis is every bit as capable of reconstructing the square step as the convergent one and every bit as capable of recovering the Gibbs phenomenon, and is thus every bit as complete. As with our earlier examples then, we once again confirm that completeness is not at all tied to normalizability.

8. Final comments

In this work, we have shown that in and of itself the requirement of normalizability of basis modes is not at all needed for completeness, and that one can construct localized steps out of bases whose modes are not normalizable at all. Since the localized steps that we have constructed out of non-normalizable bases involve expansion coefficients V_n which are explicitly found to be finite, this suggests that we should be able to construct propagators involving the modes in which these modes appear as poles which have residues which are themselves finite. Thus, in sharp contrast to the situation in which propagators are built out of normalizable modes, for propagators which are built out modes of which are not normalizable,

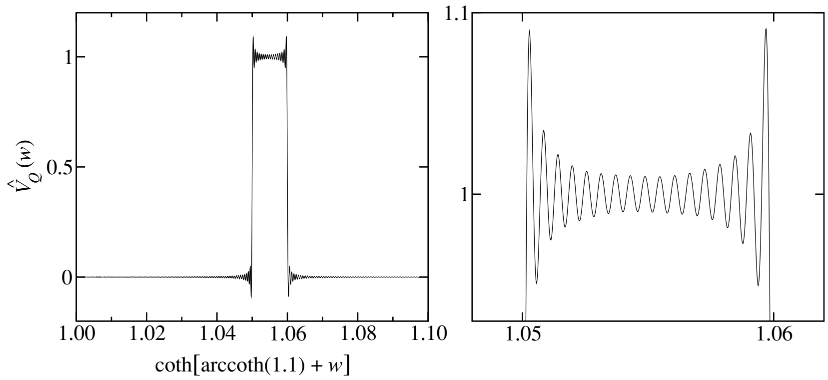

Figure 6. The left panel shows a reconstruction of the square step $\hat{V}_Q(w) = 1, 1.05 \leqslant \coth(\mathrm{arccoth}(1.1) + b|w|) \leqslant 1.06$, $\hat{V}_Q(w) = 0$ otherwise, via the dS_4^- discrete $\mathrm{Re}[Q_\nu^2(\coth(\sigma + b|w|))]$ plus $\sinh^2(\coth(\sigma+b|w|))$ mode basis in the typical case where $\coth\sigma = 1.1$, $H/b = 0.458$ and $b = 1$. The right panel shows a blow-up of the region near the top of the step.

these residues must then not be related to normalization constants or to any bilinear integrals of the modes at all for that matter.

To explicitly construct such divergent mode based propagators, we must first introduce explicit source terms. For the case of interest to the brane world, the source is typically taken to be a transverse-traceless energy–momentum tensor $S_{\mu\nu}^{\mathrm{TT}}$ which is confined to the brane at $w = 0$, with equations (10) and (11) being replaced by (see, e.g., [6])

$$\left[\frac{\partial^2}{\partial|w|^2} - 4\left(\frac{\mathrm{d}A}{\mathrm{d}|w|}\right)^2 + \mathrm{e}^{-2A}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\right]h_{\mu\nu}^{\mathrm{TT}} = 0, \tag{83}$$

$$\delta(w)\left[\frac{\partial}{\partial|w|} - 2\frac{\mathrm{d}A}{\mathrm{d}|w|}\right]h_{\mu\nu}^{\mathrm{TT}} = -\kappa_5^2\delta(w)S_{\mu\nu}^{\mathrm{TT}}, \tag{84}$$

where κ_5^2 is the brane-world gravitational constant.

For the case first of the convergent warp factor M_4^+ brane world where equations (83) and (84) reduce to

$$\left[\frac{\partial^2}{\partial|w|^2} - 4b^2 + \mathrm{e}^{2b|w|}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\right]h_{\mu\nu}^{\mathrm{TT}} = 0, \tag{85}$$

$$\delta(w)\left[\frac{\partial}{\partial|w|} + 2b\right]h_{\mu\nu}^{\mathrm{TT}} = -\kappa_5^2\delta(w)S_{\mu\nu}^{\mathrm{TT}}, \tag{86}$$

on recalling that the Bessel functions obey

$$\begin{aligned}\left[\frac{\mathrm{d}}{\mathrm{d}|w|} + 2b\right]&\left[\alpha_q J_2\left(\frac{q\,\mathrm{e}^{b|w|}}{b}\right) + \beta_q Y_2\left(\frac{q\,\mathrm{e}^{b|w|}}{b}\right)\right]\\ &= q\,\mathrm{e}^{b|w|}\left[\alpha_q J_1\left(\frac{q\,\mathrm{e}^{b|w|}}{b}\right) + \beta_q Y_1\left(\frac{q\,\mathrm{e}^{b|w|}}{b}\right)\right],\end{aligned} \tag{87}$$

an explicit solution to equations (85) and (86) can readily be given, namely [9]

$$\begin{aligned}h_{\mu\nu}^{\mathrm{TT}}(x,|w|) &= -\frac{\kappa_5^2}{(2\pi)^4}\int \mathrm{d}^4x'\mathrm{d}^4p\,\mathrm{e}^{\mathrm{i}p\cdot(x-x')}\frac{[\alpha_q J_2(q\,\mathrm{e}^{b|w|}/b) + \beta_q Y_2(q\,\mathrm{e}^{b|w|}/b)]}{q[\alpha_q J_1(q/b) + \beta_q Y_1(q/b)]}S_{\mu\nu}^{\mathrm{TT}}(x')\\ &= -2\kappa_5^2\int \mathrm{d}^4x'\hat{G}^{\mathrm{TT}}(x,x',w,0;\alpha_q,\beta_q,M_4^+)S_{\mu\nu}^{\mathrm{TT}}(x'),\end{aligned} \tag{88}$$

where $q^2 = (p^0)^2 - \bar{p}^2$ (q being understood to have the same sign as p^0 here), and α_q and β_q are arbitrary constants.

The generalization of this solution to the divergent warp factor M_4^- brane world where we have

$$\left[\frac{\partial^2}{\partial|w|^2} - 4b^2 + \mathrm{e}^{-2b|w|}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\right]h_{\mu\nu}^{\mathrm{TT}} = 0, \tag{89}$$

$$\delta(w)\left[\frac{\partial}{\partial|w|} - 2b\right]h_{\mu\nu}^{\mathrm{TT}} = -\kappa_5^2\delta(w)S_{\mu\nu}^{\mathrm{TT}}, \tag{90}$$

and

$$\begin{aligned}\left[\frac{\mathrm{d}}{\mathrm{d}|w|} - 2b\right]&\left[\alpha_q J_2\left(\frac{q\,\mathrm{e}^{-b|w|}}{b}\right) + \beta_q Y_2\left(\frac{q\,\mathrm{e}^{-b|w|}}{b}\right)\right]\\ &= -q\,\mathrm{e}^{-b|w|}\left[\alpha_q J_1\left(\frac{q\,\mathrm{e}^{-b|w|}}{b}\right) + \beta_q Y_1\left(\frac{q\,\mathrm{e}^{-b|w|}}{b}\right)\right],\end{aligned} \tag{91}$$

is of the form [6]

$$\begin{aligned}h_{\mu\nu}^{\mathrm{TT}}(x,|w|) &= \frac{\kappa_5^2}{(2\pi)^4}\int \mathrm{d}^4x'\mathrm{d}^4p\,\mathrm{e}^{\mathrm{i}p\cdot(x-x')}\frac{[\alpha_q J_2(q\,\mathrm{e}^{-b|w|}/b) + \beta_q Y_2(q\,\mathrm{e}^{-b|w|}/b)]}{q[\alpha_q J_1(q/b) + \beta_q Y_1(q/b)]}S_{\mu\nu}^{\mathrm{TT}}(x')\\ &= -2\kappa_5^2\int \mathrm{d}^4x'\hat{G}^{\mathrm{TT}}(x,x',w,0;\alpha_q,\beta_q,M_4^-)S_{\mu\nu}^{\mathrm{TT}}(x'),\end{aligned} \tag{92}$$

with α_q and β_q again being arbitrary constants. That the solution of equation (92) satisfies equation (89) follows directly, since both $J_2(q\,\mathrm{e}^{-b|w|}/b)$ and $Y_2(q\,\mathrm{e}^{-b|w|}/b)$ separately satisfy the Bessel function equation given as equation (19) with y being given by $y = q\,\mathrm{e}^{-b|w|}/b$; and that the solution satisfies equation (90) follows from equation (91). For this solution we note that it is the requirement that equation (92) obey equation (90) (technically the Israel junction condition in the presence of the source) which fixes the overall normalization of the integrand in equation (92), with none of the α_q or β_q coefficients needing to be infinite. In fact the same is true of the M_4^+ brane world propagator as its overall normalization is fixed by the junction condition of equation (86), with the similarity of the M_4^+ solution of equation (88) and the M_4^- solution of equation (92) essentially showing complete insensitivity to the normalizability or lack thereof of basis modes.

In order to be able to make contact with the various bases we used in our construction of localized steps in the divergent warp factor M_4^- brane world, we need to make specific choices for the α_q and β_q coefficients which appear in equation (92). To make contact with the convergent $J_2(q\,\mathrm{e}^{-b|w|}/b)$ modes, we recall that a Taylor series expansion of $J_1(q/b)$ around any j_i zero of J_1 is of the form

$$J_1(q/b) = \left(\frac{q}{b} - j_i\right)J_1'(j_i) = \left(\frac{q}{b} - j_i\right)\left[\frac{J_1(j_i)}{j_i} - J_2(j_i)\right] = -\left(\frac{q}{b} - j_i\right)J_2(j_i). \tag{93}$$

Thus on setting $\alpha_q = 1, \beta_q = 0$ and recalling that each j_i zero of $J_1(j_i)$ is also a zero of $J_1(-j_i)$, we see that the propagator of equation (92) contains a set of isolated poles at the zeros of J_1 (a pole at $q = bj_1$ when p^0 is positive and a pole at $q = -bj_1$ when p^0 is negative), with a p^0 plane contour integration yielding a net pole contribution to the propagator of the form

$$\hat{G}^{\mathrm{TT}}(x,0,w,0;\alpha_q = 1,\beta_q = 0,M_4^-) = -\mathrm{i}\sum_i f_i(|w|)f_i(0)\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}\bar{p}\cdot\bar{x}}}{2E_i}[\mathrm{e}^{-\mathrm{i}E_it} - \mathrm{e}^{\mathrm{i}E_it}], \tag{94}$$

where

$$f_i(|w|) = \frac{b^{1/2} J_2(j_i \mathrm{e}^{-b|w|})}{J_2(j_i)}, \qquad E_i = \left(\bar{p}^2 + b^2 j_i^2\right)^{1/2}, \tag{95}$$

and where the summation in equation (94) only needs extend over the $j_i > 0$ modes. Finally, recalling equation (32), namely

$$\int_0^\infty \mathrm{d}|w|\, \mathrm{e}^{-2b|w|} J_2^2(m\, \mathrm{e}^{-b|w|}/b) = \frac{J_2^2(m/b)}{2b}, \tag{96}$$

we see that the $f_i(|w|)$ basis modes precisely obey equations (1) and (5), with the pole structure of the M_4^- brane-world propagator $\hat{G}^{\mathrm{TT}}(x, 0, w, 0; \alpha_q = 1, \beta_q = 0, M_4^-)$ nicely recovering the orthonormality and closure structure of the normalizable $J_2^2(m\, \mathrm{e}^{-b|w|}/b)$ sector basis modes.

In order to make contact with the non-normalizable M_4^- mode sector, we need to take β_q to be non-zero in the M_4^- propagator. Recalling that $J_1(y)$, $J_2(y)$, $Y_1(y)$ and $Y_2(y)$ respectively behave as $y/2$, $y^2/8$, $-2/\pi y + O(y)$ and $-4/\pi y^2 - 1/\pi$ near $y = 0$, we see that once β_q is non-zero, the integrand $[\alpha_q J_2(q\, \mathrm{e}^{-b|w|}/b) + \beta_q Y_2(q\, \mathrm{e}^{-b|w|}/b)]/q[\alpha_q J_1(q/b) + \beta_q Y_1(q/b)]$ will behave as $2b\, \mathrm{e}^{2b|w|}/q^2$ near $q^2 = 0$ independent of the actual values of α_q and β_q, to thus give rise to a massless graviton pole term contribution of the form

$$\hat{G}^{\mathrm{TT}}(x, 0, w, 0; \alpha_q, \beta_q \neq 0, M_4^-, \text{graviton}) = \mathrm{i}b\, \mathrm{e}^{2b|w|} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\mathrm{e}^{\mathrm{i}\bar{p}\cdot\bar{x}}}{2|p|} [\mathrm{e}^{-\mathrm{i}|p|t} - \mathrm{e}^{\mathrm{i}|p|t}]. \tag{97}$$

Non-normalizable as the M_4^- brane-world graviton might be, as we see, it nonetheless appears in the propagator with a finite residue[16].

To make contact with the M_4^- brane world divergent $Y_2(q\, \mathrm{e}^{-b|w|}/b)$ modes we set $\alpha_q = 0$ in $\hat{G}^{\mathrm{TT}}(x, 0, w, 0; \alpha_q, \beta_q, M_4^-)$, and while we immediately then obtain poles at the zeros of $Y_1(q/b)$, since both $Y_2(q\, \mathrm{e}^{-b|w|}/b)$ and $Y_1(q/b)$ have branch points at $q = 0$, we also obtain a cut discontinuity, with the full singular term evaluating to [6]

$$\begin{aligned}
\hat{G}^{\mathrm{TT}}(x, 0, w, 0; \alpha_q = 0, \beta_q \neq 0, M_4^-) &= \mathrm{i}b\, \mathrm{e}^{2b|w|} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\mathrm{e}^{\mathrm{i}\bar{p}\cdot\bar{x}}}{2|p|} [\mathrm{e}^{-\mathrm{i}|p|t} - \mathrm{e}^{\mathrm{i}|p|t}] \\
&\quad - \mathrm{i} \sum_i \tilde{f}_i(|w|) \tilde{f}_i(0) \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\mathrm{e}^{\mathrm{i}\bar{p}\cdot\bar{x}}}{2E_i} [\mathrm{e}^{-\mathrm{i}E_i t} - \mathrm{e}^{\mathrm{i}E_i t}] \\
&\quad + \frac{\mathrm{i}}{(2\pi)^3} \int \mathrm{d}^3 p \frac{\mathrm{e}^{\mathrm{i}\bar{p}\cdot\bar{x}}}{2E_p} [\mathrm{e}^{-\mathrm{i}E_p t} - \mathrm{e}^{\mathrm{i}E_p t}] \int dm \left[1 - 2i \frac{J_2(m\, \mathrm{e}^{-b|w|}/b)}{Y_1(m/b)}\right] \\
&\quad \times \left[\frac{[Y_1(m/b) J_2(m\, \mathrm{e}^{-b|w|}/b) - J_1(m/b) Y_2(m\, \mathrm{e}^{-b|w|}/b)]}{\pi\left[4J_1^2(m/b) + Y_1^2(mb)\right]}\right],
\end{aligned} \tag{98}$$

where

$$\tilde{f}_i(|w|) = \frac{b^{1/2} Y_2(y_i\, \mathrm{e}^{-b|w|})}{Y_2(y_i)}, \qquad E_i = \left(\bar{p}^2 + b^2 y_i^2\right)^{1/2}. \tag{99}$$

As we again see, despite the lack of normalizability of $Y_2(m\, \mathrm{e}^{-b|w|}/b)$ modes, all the terms which appear in equation (98) do so with coefficients which are nonetheless finite.

[16] Despite the fact that the negative tension M_4^- brane world possesses a massless graviton whose residue is finite, we note that its residue appears with an overall minus sign (namely negative signature) compared to the otherwise identical in structure positive signature massless graviton residue of the positive tension M_4^+ brane world (compare the first forms given for $h_{\mu\nu}^{\mathrm{TT}}(x, |w|)$ given in equations (88) and (92) which differ by an overall minus sign occasioned by the overall difference in sign between the right-hand sides of equations (87) and (91)). Such negative signature is thought to indicate an instability of the M_4^- brane world. Nonetheless, even though the M_4^- brane world might thus not be of direct physical interest, it can still serve as a useful mathematical laboratory for exploring the completeness properties of bases built out of non-normalizable modes.

Further examples of this phenomenon may be found in the other divergent warp factor brane worlds we have been considering. However, unlike the exact propagator solutions of equations (88) and (92), for the AdS_4 and dS_4 based brane worlds so far such propagators have only been constructed in low order. Specifically, for the AdS_4^+ brane world for instance where the background metric of equation (6) takes the explicit form

$$\mathrm{d}s^2 = \mathrm{d}w^2 + \mathrm{e}^{2A(|w|)}[\mathrm{d}x^2 + \mathrm{e}^{2Hx}(\mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}t^2)] \tag{100}$$

with the AdS_4^+ warp factor $A(|w|)$ being given in equation (9) and λ being positive, to lowest order in H the appropriate AdS_4^+ propagator is given as [6]

$$\begin{aligned}\hat{G}^{\mathrm{TT}}\big(x, x', w, 0; \hat{\alpha}_\nu, \hat{\beta}_\nu, \mathrm{AdS}_4^+\big) &= \frac{1}{2H(2\pi)^3}\int_{-\infty}^{\infty}\mathrm{d}p^0\mathrm{d}p^2\mathrm{d}p^3\int_0^{\infty}\mathrm{d}p^1\,p^1 B_\nu(\tanh(b|w|-\sigma), \hat{\alpha}_\nu, \hat{\beta}_\nu)\\ &\quad\times \mathrm{e}^{Hx/2}\,\mathrm{e}^{Hx'/2}\,\mathrm{e}^{-\mathrm{i}p^0(t-t')+\mathrm{i}p^2(y-y')+\mathrm{i}p^3(z-z')}J_\tau(k\,\mathrm{e}^{-Hx}/H)J_\tau(k\,\mathrm{e}^{-Hx'}/H)\end{aligned} \tag{101}$$

where k is given by $k = [(p^0)^2 - (p^2)^2 - (p^3)^2]^{1/2}$, τ and ν are given by $\tau = \nu + 1/2 = [9/4 + k^2/H^2 - (p^1)^2/H^2]^{1/2}$, and the quantity $B_\nu(\tanh(b|w| - \sigma), \hat{\alpha}_\nu, \hat{\beta}_\nu)$ is given by

$$\begin{aligned}B_\nu(\tanh(b|w|-\sigma), \hat{\alpha}_\nu, \hat{\beta}_\nu) &= \frac{1}{H(\nu-1)(\nu+2)}\left[\frac{\hat{\alpha}_\nu P_\nu^2(\tanh(b|w|-\sigma)) + \hat{\beta}_\nu Q_\nu^2(\tanh(b|w|-\sigma))}{\hat{\alpha}_\nu P_\nu^1(-\tanh\sigma) + \hat{\beta}_\nu Q_\nu^1(-\tanh\sigma)}\right].\end{aligned} \tag{102}$$

As constructed the quantity $B_\nu(\tanh(b|w| - \sigma), \hat{\alpha}_\nu, \hat{\beta}_\nu)$ obeys

$$\delta(w)\left[\frac{\mathrm{d}}{\mathrm{d}|w|} - 2\frac{\mathrm{d}A}{\mathrm{d}|w|}\right]B_\nu(\tanh(b|w|-\sigma), \hat{\alpha}_\nu, \hat{\beta}_\nu) = \delta(w), \tag{103}$$

and has a small H limit into the analogous M_4^+ integrand, namely

$$B_\nu(\tanh(b|w|-\sigma), \hat{\alpha}_\nu, \hat{\beta}_\nu) \to \frac{[\alpha_q J_2(q\,\mathrm{e}^{b|w|}/b) + \beta_q Y_2(q\,\mathrm{e}^{b|w|}/b)]}{q[\alpha_q J_1(q/b) + \beta_q Y_1(q/b)]}, \tag{104}$$

where $\hat{\alpha}_\nu = \alpha_q\cos(\nu\pi) + \beta_q\sin(\nu\pi)$, $\hat{\beta}_\nu = (2/\pi)[-\alpha_q\sin(\nu\pi) + \beta_q\cos(\nu\pi)]$. In the small H limit $\hat{G}^{\mathrm{TT}}(x, x', w, 0; \hat{\alpha}_\nu, \hat{\beta}_\nu, \mathrm{AdS}_4^+)$ obeys

$$\begin{aligned}\left[\frac{\partial^2}{\partial w^2} - 4\left(\frac{\mathrm{d}A}{\mathrm{d}|w|}\right)^2 - 4\frac{\mathrm{d}A}{\mathrm{d}|w|}\delta(w) + \mathrm{e}^{-2A}\tilde{\nabla}_\alpha\tilde{\nabla}^\alpha\right]\hat{G}^{\mathrm{TT}}\big(x, x', w, 0; \hat{\alpha}_\nu, \hat{\beta}_\nu, \mathrm{AdS}_4^+\big)\\ = \mathrm{e}^{Hx}\delta(x-x')\delta(t-t')\delta(y-y')\delta(z-z')\delta(w),\end{aligned} \tag{105}$$

with the fluctuation

$$h_{\mu\nu}^{\mathrm{TT}}(x, |w|) = -2\kappa_5^2\int \mathrm{d}^4x'\mathrm{e}^{-Hx'}\hat{G}^{\mathrm{TT}}\big(x, x', w, 0; \hat{\alpha}_\nu, \hat{\beta}_\nu, \mathrm{AdS}_4^+\big)S_{\mu\nu}^{\mathrm{TT}}(x') \tag{106}$$

thus being an exact AdS_4^+ brane world small H solution to the AdS_4^+ variant of equations (83) and (84) for an arbitrary $S_{\mu\nu}^{\mathrm{TT}}(x')$ source on the brane.

As regards pole terms in the AdS_4^+ brane-world propagator, since $(\nu-1)(\nu+2) = q^2/H^2$, the $(\nu - 1)(\nu + 2)$ term in $B_\nu(\tanh(b|w| - \sigma), \hat{\alpha}_\nu, \hat{\beta}_\nu)$ generates a massless $\nu = 1$ graviton pole in the propagator which is found to be of the form [6]

$$\hat{G}^{\mathrm{TT}}\big(x, x', w, 0; \hat{\alpha}_\nu, \hat{\beta}_\nu, \mathrm{AdS}_4^+, \text{graviton}\big) = \frac{b\,\mathrm{e}^{2A}\hat{D}_S(x, x', m=0)}{[-(\hat{\alpha}_1/\hat{\beta}_1)(H^2/b^2) + (1 - H^2/b^2)^{1/2} + (H^2/b^2)\mathrm{arccosh}(b/H)]}, \tag{107}$$

where $\hat{D}_S(x, x', m)$ is the pure AdS_4 spacetime propagator which obeys

$$\left[\partial_x^2 - H\partial_x + e^{-2Hx}\left(\partial_y^2 + \partial_z^2 - \partial_t^2\right) - 2H^2 - m^2\right]\hat{D}_S(x, x', m) = e^{Hx}\delta^4(x - x'). \quad (108)$$

As we see, despite the lack of normalizability of the graviton wavefunction, the residue at the AdS_4^+ massless graviton pole is nonetheless finite[17]. Similarly, if we set $\hat{\alpha}_\nu = 0$ in equation (101) we will immediately generate the divergent $Q_\nu^2(\tanh(b|w| - \sigma))$ modes as poles associated with the zeros of $Q_\nu^1(-\tanh\sigma)$, with these pole terms also possessing finite residues. Consequently, in the brane world divergent modes are fully capable of appearing with finite residues in propagators and their lack of normalizability should not be taken as being a criterion for excluding them. In fact, with the M_4^+ propagator $\hat{G}^{TT}(x, x', w, 0; \alpha_q, \beta_q, M_4^+)$ of equation (88) being causal when we set $\alpha_q = 1, \beta_q = i$ [9, 10] (so that it is then based on outgoing Hankel functions), given the small H limit of $B_\nu(\tanh(b|w| - \sigma), \hat{\alpha}_\nu, \hat{\beta}_\nu)$ exhibited in equation (104), it will be the $\hat{G}^{TT}(x, x', w, 0; \hat{\alpha}_\nu, \hat{\beta}_\nu, AdS_4^+)$ propagator with $\hat{\alpha}_\nu = e^{i\pi\nu}, \hat{\beta}_\nu = (2i/\pi)\, e^{i\pi\nu}$ which will be the AdS_4^+ analogue of the outgoing Hankel function based causal M_4^+ brane-world propagator, with this particular AdS_4^+ brane world propagator explicitly being found to be causal [6]. As such, the causal AdS_4^+ brane-world propagator with $\hat{\alpha}_\nu = e^{i\pi\nu}, \hat{\beta}_\nu = (2i/\pi)\, e^{i\pi\nu}$ possesses an explicit massless graviton pole whose residue is finite, with there thus being no justification for excluding it[18].

Acknowledgments

The authors would like to thank Dr A H Guth, Dr D I Kaiser and Dr A Nayeri for their active participation in this work, and for their many helpful comments.

References

[1] Bender C M, Brody D C and Jones H F 2003 *Am. J. Phys.* **71** 1095
Mannheim P D and Davidson A 2005 *Phys. Rev.* A **71** 042110
[2] Randall L and Sundrum R 1999 *Phys. Rev. Lett.* **83** 3370
[3] Randall L and Sundrum R 1999 *Phys. Rev. Lett.* **83** 4690
[4] DeWolfe O, Freedman D Z, Gubser S S and Karch A 2000 *Phys. Rev.* D **62** 046008
[5] Kim H B and Kim H D 2000 *Phys. Rev.* D **61** 064003
[6] Mannheim P D 2005 *Brane-Localized Gravity* (Hackensack, NJ: World Scientific)
[7] Karch A and Randall L 2001 *J. High Energy Phys.* JHEP05(2001)008
[8] Garriga J and Sasaki M 2000 *Phys. Rev.* D **62** 043523
[9] Giddings S B, Katz E and Randall L 2000 *J. High Energy Phys.* JHEP03(2000)023
[10] Mannheim P D 2006 Causality in the brane world *Preprint* hep-th/0607041

[17] Unlike the massless graviton of the negative tension M_4^- brane world, the massless graviton of the positive tension AdS_4^+ brane world has a residue with a perfectly acceptable positive signature—as indeed it must since its residue continues into that of the positive signature massless graviton of the positive tension M_4^+ brane world in the limit in which H is taken to zero.

[18] From the perspective of the possible physical viability of the AdS_4^+ brane world, the need to include non-normalizable modes is actually somewhat unfortunate since they lead to a gravity which is not at all localized to the brane. It was the fact that a restriction to normalized modes did lead to gravitational fluctuation modes which were localized to the brane which prompted the AdS_4^+ study of [7].